

Quasitriangular WZW model

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Abstract

A dynamical system is canonically associated to every Drinfeld double of any affine Kac-Moody group. The choice of the affine Lu-Weinstein-Soibelman double gives a smooth one-parameter deformation of the standard WZW model. In particular, the worldsheet and the target of the classical version of the deformed theory are the ordinary smooth manifolds. The quasitriangular WZW model is exactly solvable and it admits the chiral decomposition. Its classical action is not invariant with respect to the left and right action of the loop group, however it satisfies the weaker condition of the Poisson-Lie symmetry. The structure of the deformed WZW model is characterized by several ordinary and dynamical r -matrices with spectral parameter. They describe the q -deformed current algebras, they enter the definition of q -primary fields and they characterize the quasitriangular exchange (braiding) relations. Remarkably, the symplectic structure of the deformed chiral WZW theory is cocharacterized by the same elliptic dynamical r -matrix that appears in the Bernard generalization of the Knizhnik-Zamolodchikov equation, with q entering the modular parameter of the Jacobi theta functions. This reveals a tantalizing connection between the classical q -deformed WZW model and the quantum standard WZW theory on elliptic curves and opens the way for the systematic use of the dynamical Hopf algebroids in the rational q -conformal field theory.

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Chapter 1

Introduction

1. Basic observation. The WZW model [47] is certainly one of the most important models of the two-dimensional (conformal) field theory. It is well-known that many interesting theories can be naturally obtained by its reductions, like e.g. the coset models [31] or Toda theories [22]. Such fabrication of new structures from the roof model was called "the WZW factory" in [28].

This paper is based on an observation, that the roof of the WZW factory is in fact two floors above the WZW model. In other words, we shall argue that there is a master model from which the WZW model can be obtained by two successive symplectic reductions. This master model describes the geodesical flow on the affine Kac-Moody group $\tilde{G} = \mathbf{R} \times_{\hat{s}} \hat{G}$ and its action reads

$$S(\tilde{g}) = -\frac{\kappa}{4} \int d\tau \left(\tilde{g}^{-1} \frac{d}{d\tau} \tilde{g}, \tilde{g}^{-1} \frac{d}{d\tau} \tilde{g} \right)_{\tilde{G}}. \quad (1.1)$$

Here $\tilde{g}(\tau) \in \tilde{G}$, κ is going to play the role of the WZW level, $(\cdot, \cdot)_{\tilde{G}}$ is the invariant inner product on $\tilde{\mathcal{G}} = Lie(\tilde{G})$, $\hat{G} = \widehat{LG}_0$ is the centrally extended loop group and $\times_{\hat{s}}$ means the semidirect product corresponding to the loop group parameter shift automorphism $\sigma \rightarrow \sigma + s$.

The reader may be surprised that neither the world-sheet space derivative ∂_σ nor the WZW term are present in the master action (1.1). We shall see, however, that they are "born" under the process of the symplectic reduction.

The interest in lifting the WZW model relies on the fact that the master model sitting two floors higher has extremely simple structure. Its phase space is the cotangent bundle $T^*\tilde{G}$ equipped with its canonical symplectic structure. We can therefore easily construct deformations of the master

model by using the theory of various doubles (Manin, Drinfeld, Heisenberg) of the group \tilde{G} . One simply replaces the cotangent bundle $T^*\tilde{G}$ by a chosen Drinfeld double \tilde{D} and the symplectic structure is then canonically given by the Semenov-Tian-Shansky two-form $\tilde{\omega}$ on \tilde{D} . The left-right \tilde{G} -symmetries of the master model (1.1) get thus deformed to Poisson-Lie symmetries, and the Hamiltonian charges (=Abelian moment maps generating the standard symmetries) become non-Abelian Poisson-Lie moment maps. We then obtain the quasitriangular WZW theory by performing the two step symplectic reduction of the deformed master model.

Remark: It is also interesting to note that the various symplectic reductions can be applied also in the Poisson-Lie case, or, in other words, the whole WZW factory should survive the q -deformation.

2. Chiral decomposition. It is well-known that the standard WZW model can be obtained by the appropriate glueing of two identical copies of a simpler dynamical system called the chiral WZW model [20]. The same thing turns out to be true also for the master model (1.1). It can be glued up from two copies of the chiral geodesical model whose (first order Hamiltonian) action is given by

$$\tilde{S}_L(\tilde{k}, \tilde{\phi}) = \int d\tau [\langle \tilde{\phi}, \tilde{k}^{-1} \frac{d}{d\tau} \tilde{k} \rangle + \frac{1}{2\kappa} (\tilde{\phi}, \tilde{\phi})_{\tilde{G}^*}], \quad (1.2)$$

where $\tilde{k}(\tau) \in \tilde{G}$ and $\tilde{\phi}(\tau) \in \tilde{\mathcal{A}}_+$. Note that $\tilde{\mathcal{A}}_+$ is the Weyl alcove viewed as the subset of the dual of the Cartan subalgebra $\tilde{\mathcal{T}}$ of $\tilde{\mathcal{G}}$.

We shall show that the chiral WZW model can be also obtained by a simple two-step symplectic reduction from the chiral master model (1.2). In fact, the σ -shift and the central circle subgroups of \tilde{G} act in the standard Hamiltonian way on the phase space $\tilde{M}_L \equiv \tilde{G} \times \tilde{\mathcal{A}}_+$. The reduction is then induced by setting the σ -shift Hamiltonian charge to 0 and its central circle fellow to κ .

It turns out that the master model (1.1) and the chiral geodesical model (1.2) have natural deformations based on the choice of an appropriate Drinfeld double \tilde{D} of the affine Kac-Moody group \tilde{G} . In particular, the resulting deformed chiral geodesical model is formulated on the same phase space \tilde{M}_L

but now its action reads

$$\tilde{S}_L^q(\tilde{k}, \tilde{\phi}) = \int [\theta^q + \frac{1}{2\kappa}(\tilde{\phi}, \tilde{\phi})_{\tilde{G}^*} d\tau]. \quad (1.3)$$

In order to explain the notation: There exists a one-parameter family of embeddings of the affine model space \tilde{M}_L into the Drinfeld double $\tilde{\tilde{D}}$. This parameter will be referred to as $q \equiv e^\varepsilon$. Then θ^q is the solution¹ of the equation $d\theta^q = \Omega^q$, where Ω^q is the pull-back of the Semenov-Tian-Shansky form to the q -embedded submanifold $\tilde{M}_L \hookrightarrow \tilde{\tilde{D}}$.

There is the crucial condition to be imposed on $\tilde{\tilde{D}}$, namely, the σ -shift and the central circle subgroups of \tilde{G} must still act on \tilde{M}_L in the standard Hamiltonian way but now with respect to the symplectic structure Ω^q . Such good doubles will be referred to as the WZW doubles of \tilde{G} . As in the non-deformed case, the quasitriangular chiral WZW model will be then obtained from the action (1.3) by the κ -depending symplectic reduction based on setting the corresponding σ -shift and central circle Hamiltonian charges to 0 and κ , respectively.

Although the σ -shift and the central circle still act in the standard Hamiltonian way on (\tilde{M}_L, Ω^q) , this is non longer true for the action of the remaining loop group generators of $Lie(\tilde{G})$. Nevertheless, due to the fact that Ω^q is the pull-back of the Semenov-Tian-Shansky form, the remaining generators act in the Poisson-Lie way. This means, in particular, that the action (1.3) of the deformed chiral geodesical model is Kac-Moody Poisson-Lie symmetric. Due to this property, the quantized quasitriangular chiral WZW model will enjoy the q -Kac-Moody symmetry.

Remarks : i) The deformation of the master model (1) exists for every Drinfeld double of \tilde{G} . However, the two-step symplectic reduction can be performed only if $\tilde{\tilde{D}}$ is the WZW double (see above). We found with satisfaction that a nontrivial WZW double $\tilde{\tilde{D}}$ of \tilde{G} can be indeed constructed; it is in fact nothing but the complexification $\tilde{\tilde{D}} = \tilde{G}^{\mathbb{C}}$ of the affine Kac-Moody group \tilde{G} equipped with certain invariant maximally noncompact inner product on $\tilde{\tilde{D}} = Lie(\tilde{\tilde{D}})$. We shall call $\tilde{\tilde{D}}$ the "affine Lu-Weinstein-Soibelman double", since it turns out to be the natural affine generalization of the standard Lu-Weinstein-Soibelman double D_0 of the group G_0 .

¹The classical action (1.3) makes sense even if this solution θ^q exists only locally on \tilde{M}_L .

ii) The Weyl alcove \mathcal{A}_+ is usually viewed as the fundamental domain of the action of the affine Weyl group on the Cartan subalgebra \mathcal{T} of $\mathcal{G}_0 \equiv Lie(G_0)$. The affine Weyl group acts also on the Cartan subalgebra $\tilde{\mathcal{T}}$ of $\tilde{\mathcal{G}} \equiv Lie(\tilde{G})$. The alcove $\tilde{\mathcal{A}}_+$ is again the fundamental domain of this action. It is also important to note that the second floor chiral Hamiltonian

$$\tilde{H}_L = -\frac{1}{2\kappa}(\tilde{\phi}, \tilde{\phi})_{\tilde{\mathcal{G}}^*} \quad (1.4)$$

does not depend on q . The q -dependence of its descendant H_L^{qWZ} will turn out to be the fruit of the reduction. The reader will find more detailed explanations of all that in the body of the paper.

iii) The q -deformations of the WZW model have been already studied in the literature [20, 12, 17]. However, in those cases either the worldsheet or the target of the σ -model was first *kinematically* deformed to become a lattice or a non-commutative manifold. Then a kind of a (discrete or non-commutative) dynamics was formulated on this deformed background. What we are doing here is somewhat different; we avoid any preliminary kinematical deformation of the worldsheet or of the target. The q -deformed objects are generated *dynamically*. This means that they appear solely as the result of standard field theoretical quantization of some (chiral) classical theory whose phase space M_L^{WZ} is topologically *the same* as that of the non-deformed chiral WZW theory. The things that get (smoothly) deformed are the symplectic form and the Hamiltonian function on this unchanged phase space.

3. Quasitriangular symplectic structure. As already stated in the remark iii) above, the crucial result of the two-step symplectic reduction of (1.3) is the fact that the phase space of the quasitriangular chiral WZW model is topologically *the same* manifold as the phase space M_L^{WZ} of the non-deformed standard chiral WZW theory. Recall that points in M_L^{WZ} are the maps $m : \mathbf{R} \rightarrow G_0$, fulfilling the monodromy condition

$$m(\sigma + 2\pi) = m(\sigma)M. \quad (1.5)$$

Here the monodromy² $M = \exp(-2\pi i a^\mu H^\mu)$ sits in the fundamental Weyl alcove viewed as the subset of the maximal torus \mathbf{T} of G_0 . In what follows,

²Sometimes people consider [29, 20, 6] the bigger chiral WZW phase space in the sense that M can be an arbitrary element of G_0 . Such an enlargement is useful for description of the (finite dimensional) quantum group symmetries of the standard WZW model, however,

a^μ will be coordinates on the alcove \mathcal{A}_+ corresponding to the choice of the orthonormal basis H^μ on the Cartan subalgebra.

Although the phase space M_L^{WZ} is the same, the symplectic structure ω_L^{qWZ} and the Hamiltonian H_L^{qWZ} differ, however, from their non-deformed WZW counterparts. One of the main results of this paper is the explicit description of the pair $(\omega_L^{qWZ}, H_L^{qWZ})$. Thus the symplectic structure corresponding to the two-form ω_L^{qWZ} is fully characterized by the following Poisson bracket

$$\{m(\sigma) \otimes m(\sigma')\}_{qWZ} = (m(\sigma) \otimes m(\sigma')) B_\varepsilon(a^\mu, \sigma - \sigma') + \varepsilon \hat{r}(\sigma - \sigma') (m(\sigma) \otimes m(\sigma')), \quad (1.6)$$

where $B_\varepsilon(a^\mu, \sigma)$ is the so called quasitriangular braiding matrix given by

$$B_\varepsilon(a^\mu, \sigma) = -\frac{i}{\kappa} \rho\left(\frac{i\sigma}{2\kappa\varepsilon}, \frac{i\pi}{\kappa\varepsilon}\right) H^\mu \otimes H^\mu - \frac{i}{\kappa} \sum_{\alpha \in \Phi} \frac{|\alpha|^2}{2} \sigma_{a^\mu \langle \alpha, H^\mu \rangle} \left(\frac{i\sigma}{2\kappa\varepsilon}, \frac{i\pi}{\kappa\varepsilon}\right) E^\alpha \otimes E^{-\alpha} \quad (1.7)$$

and $\hat{r}(\sigma)$ is defined as

$$\hat{r}(\sigma) = r + C \cotg \frac{1}{2} \sigma. \quad (1.8)$$

Here r and C are the ordinary (non-affine) r -matrix and Casimir elements, respectively, given by

$$r = \sum_{\alpha \in \Phi_+} \frac{i|\alpha|^2}{2} (E^{-\alpha} \otimes E^\alpha - E^\alpha \otimes E^{-\alpha}); \quad (1.9)$$

$$C = \sum_{\mu} H^\mu \otimes H^\mu + \sum_{\alpha \in \Phi_+} \frac{|\alpha|^2}{2} (E^{-\alpha} \otimes E^\alpha + E^\alpha \otimes E^{-\alpha}). \quad (1.10)$$

The functions $\rho(z, \tau), \sigma_w(z, \tau)$ are defined as (cf. [21, 22, 18])

$$\sigma_w(z, \tau) = \frac{\theta_1(w - z, \tau) \theta'_1(0, \tau)}{\theta_1(w, \tau) \theta_1(z, \tau)}, \quad \rho(z, \tau) = \frac{\theta'_1(z, \tau)}{\theta_1(z, \tau)}, \quad (1.11)$$

it is not necessary for recovering the full left-right WZW model by appropriate glueing of two chiral models. The choice of the maximal torus monodromy is sufficient to do this job and we stick on it for the rest of this paper.

where $\theta_1(z, \tau)$ is the Jacobi theta function

$$\theta_1(z, \tau) = - \sum_{j=-\infty}^{\infty} e^{\pi i(j+\frac{1}{2})^2 \tau + 2\pi i(j+\frac{1}{2})(z+\frac{1}{2})}. \quad (1.12)$$

In (1.11), the apostrof ' means the derivative with respect to the first argument z , the argument τ (the modular parameter) is a nonzero complex number such that $\text{Im}\tau > 0$.

Remark : In Sections 5.1 and 5.2, we describe in detail the hard work needed to arrive from the Semenov-Tian-shansky form on the double $\tilde{\tilde{D}}$ to the q WZW chiral Poisson bracket $\{.,.\}_{qWZ}$. Inspite of the intermediate complicated calculations, the resulting formula (1.14) is simple and esthetically appealing. Tantalizingly, $B_\varepsilon(a^\mu, \sigma)$ is the Felder-Wieczerkowski elliptic dynamical r -matrix that appears in the Bernard generalization of the Knizhnik-Zamolodchikov equation [7, 21, 22]. Apart from some deeper sense lurking at us from this fact, there is yet another profitable circumstance: the corresponding quantum dynamical R -matrix is known [21]. No doubt, an important part of the structure of a q -deformed conformal field theory will be then underlied by the concept of the dynamical Hopf algebroid corresponding to R .

In the limit $\varepsilon \rightarrow 0$, Eq. (1.6) becomes

$$\{m(\sigma) \circledast m(\sigma')\}_{(q=1)WZ} = (m(\sigma) \otimes m(\sigma'))B_0(a^\mu, \sigma - \sigma'), \quad (1.13)$$

where

$$B_0(a^\mu, \sigma) = -\frac{\pi}{\kappa} \left[\eta(\sigma)(H^\mu \otimes H^\mu) - i \sum_{\alpha \in \Phi} \frac{|\alpha|^2}{2} \frac{\exp(i\pi\eta(\sigma)\langle \alpha, H^\mu \rangle a^\mu)}{\sin(\pi\langle \alpha, H^\mu \rangle a^\mu)} E^\alpha \otimes E^{-\alpha} \right]. \quad (1.14)$$

Here $\eta(\sigma)$ is the function defined by

$$\eta(\sigma) = 2\left[\frac{\sigma}{2\pi}\right] + 1, \quad (1.15)$$

where $[\sigma/2\pi]$ is the largest integer less than or equal to $\frac{\sigma}{2\pi}$. It is shown in Section 3.2, that the relation (1.13) completely characterize the symplectic structure ω_L^{WZ} of the standard non-deformed chiral WZW model.

4. Quasitriangular Hamiltonian. The Hamiltonian H_L^{qWZ} descends (upon the reduction) from the second floor master Hamiltonian (1.4). We

want to make explicit how H_L^{qWZ} is defined as the function on the phase space M_L^{WZ} . For this purpose, it is more convenient to perform the classical (inverse) vertex-IRF transformation defined by

$$k(\sigma) = m(\sigma) \exp (ia^\mu H^\mu \sigma). \quad (1.16)$$

Note that $k(\sigma)$ then becomes periodic, hence an element of the loop group $G = LG_0$. Therefore, topologically, $M_L^{WZ} = LG_0 \times \mathcal{A}_+$.

We shall first start with the case $q = 1$ that gives the standard chiral WZW Hamiltonian. Some basic knowledge of the Poisson-Lie world is needed for understanding the case $q \neq 1$. The interested reader may consult Sections 4.1 where the relevant Poisson-Lie notions are explained.

The standard chiral WZW Hamiltonian H_L^{WZ} is usually written in the monodromic variables $m(\sigma)$ and it is given by the Sugawara formula

$$H_L^{WZ}(m) = -\frac{1}{2\kappa}(\kappa\partial_\sigma mm^{-1}, \kappa\partial_\sigma mm^{-1})_{\mathcal{G}^0}. \quad (1.17)$$

The minus sign appears because in our conventions the form $(\cdot, \cdot)_{\mathcal{G}_0}$ is negative definite. In the variables (k, a^μ) , it becomes

$$H_L^{WZ}(k, a^\mu) = -\frac{1}{2\kappa}(\phi, \phi)_{\mathcal{G}^*} - \langle \phi, k^{-1}\partial_\sigma k \rangle - \frac{\kappa}{2}(k^{-1}\partial_\sigma k, k^{-1}\partial_\sigma k)_{\mathcal{G}}, \quad (1.18)$$

where $(\hat{\phi}_\kappa)' = \phi$ (see the meaning of this notation in a while) and $(\cdot, \cdot)_{\mathcal{G}}$ is the invariant scalar product on the loop group Lie algebra $\mathcal{G} = L\mathcal{G}_0$. As always in the paper $\langle \cdot, \cdot \rangle$ means the canonical pairing between the elements of mutually dual spaces. The formula (1.18) can be rewritten in the following way

$$H_L^{WZ}(k, a^\mu) = -\frac{1}{2\kappa}(\phi, \phi)_{\mathcal{G}^*} - (\widetilde{Coa}_{\hat{k}} \hat{\phi}_\kappa)^0. \quad (1.19)$$

It turns out that for generic q , (1.19) generalizes to

$$H_L^{qWZ}(k, a^\mu) = -\frac{1}{2\kappa}(\phi, \phi)_{\mathcal{G}^*} - (\widetilde{Dres}_{\hat{k}} e^{\tilde{\Lambda}(\hat{\phi}_\kappa)})^0. \quad (1.20)$$

Recall that $\hat{\phi}_\kappa$ is the function of a^μ hence the Hamiltonians really depend on the indicated variables.

Notations 1.1: i) Denote $\tilde{T}^0 \in \tilde{\mathcal{G}}$ and $\tilde{T}^\infty \in \tilde{\mathcal{G}}$ the generators of the σ -shift and of the central circle, respectively. Then we have the linear space

decomposition $\tilde{\mathcal{G}} = \mathbf{R}\tilde{T}^0 + \mathbf{R}\tilde{T}^\infty + \mathcal{G}$ and its dual $\tilde{\mathcal{G}}^* = \mathbf{R}\tilde{t}_0 + \mathbf{R}\tilde{t}_\infty + \mathcal{G}^*$. Now we define

$$\tilde{x} = \tilde{x}^0 \tilde{t}_0 + \tilde{x}^\infty \tilde{t}_\infty + \tilde{x}', \quad \tilde{x} \in \tilde{\mathcal{G}}^*, \tilde{x}' \in \mathcal{G}^*. \quad (1.21)$$

In words: \tilde{x}^0 is the σ -shift part of \tilde{x} , \tilde{x}^∞ the central circle part and \tilde{x}' the \mathcal{G}^* -part. The lifted alcove $\hat{\phi}_\kappa$ is then characterized by the relations :

$$(\hat{\phi}_\kappa)^0 = 0, \quad (\hat{\phi}_\kappa)^\infty = \kappa, \quad (\hat{\phi}_\kappa)' = \phi. \quad (1.22)$$

ii) Consider $\tilde{\mathcal{B}} = \text{Lie}(\tilde{B})$ and $\mathcal{B} = \text{Lie}(B)$, where \tilde{B}, B are respectively the dual³ Poisson-Lie groups of \tilde{G}, G . There is the following unique decomposition of any element $\tilde{b} \in \tilde{B}$:

$$\tilde{b} = \exp(\tilde{b}^0 \tilde{\Lambda}(\tilde{t}_0)) \exp(\tilde{b}^\infty \tilde{\Lambda}(\tilde{t}_\infty)) \tilde{b}', \quad \tilde{b} \in \tilde{B}, \tilde{b}' \in B, \quad (1.23)$$

where \tilde{b}' is the Poisson-Lie analogue of \tilde{x}' and the real numbers $\tilde{b}^0, \tilde{b}^\infty$ are the analogues of $\tilde{x}^0, \tilde{x}^\infty$. The map $\tilde{\Lambda} : \tilde{\mathcal{G}}^* \rightarrow \tilde{\mathcal{B}}$ is the identification map defined by the invariant bilinear form $(\cdot, \cdot)_{\tilde{D}}$ on the Drinfeld double \tilde{D} of \tilde{G} . This form can be arbitrarily normalized. This normalization parameter is actually the deformation parameter of our WZW story. We call it either ε or q , with $q = e^\varepsilon$. Note that $\tilde{\Lambda}$ then depends on q which gives the q -dependence of the quasitriangular Hamiltonian (1.20).

iii) \widetilde{Coad} means the \tilde{G} -coadjoint action on $\hat{\phi}_\kappa$ viewed as the element of $\tilde{\mathcal{G}}^*$ and \widetilde{Dres} means the \tilde{G} -dressing action on $e^{\tilde{\Lambda}(\hat{\phi}_\kappa)}$ viewed as the element of \tilde{B} .

iv) $\hat{G} = \widehat{LG}_0$ is the principal $U(1)$ -bundle over $G = LG_0$ with the projection π . Then \hat{k} is any element of \hat{G} such that $\pi(\hat{k}) = k$. Since $\tilde{G} = \mathbf{R} \times_{\hat{S}} \hat{G}$, we can view \hat{k} also as the element of \tilde{G} and it is in this sense that \hat{k} appears in (1.19) and (1.20).

It is shown in section 5.2.7, that the $q \rightarrow 1$ limit of H_L^{qWZ} gives indeed H_L^{WZ} . We have already learned that the symplectic form ω_L^{qWZ} has also the correct $q = 1$ limit. Thus we conclude that the quasitriangular chiral WZW model is indeed the smooth deformation of its standard counterpart.

³The existence and the properties of \tilde{B}, B are implied by the structure of the Drinfeld double \tilde{D} .

5. Quasitriangular classical action. We have just described explicitly the pair $(\omega_L^{qWZ}, H_L^{qWZ})$. Knowing these data, we can write down the following classical action of the deformed chiral WZW model

$$S_L^{qWZ}[\eta_L(\tau)] = \int (\eta_L^* \theta_L^{qWZ} - H_L^{qWZ}(\eta_L) d\tau) \quad (1.24)$$

Here $\eta_L(\tau)$ is a trajectory in the phase space M_L^{WZ} parametrized by the ordinary (continuous) time parameter τ , θ_L^{qWZ} is a 1-form on the phase space called the symplectic potential and $\eta_L^* \theta_L^{qWZ}$ is its pullback by the map η_L . The symplectic form ω_L^{qWZ} on M_L^{WZ} can be then written as

$$\omega_L^{qWZ} = d\theta_L^{qWZ}. \quad (1.25)$$

Consider manifolds $M_L^{WZ} = LG_0 \times \mathcal{A}_+$ and $M_R^{WZ} = LG_0 \times \mathcal{A}_-$. Here $\mathcal{A}_- = -\mathcal{A}_+$. The full left-right quasitriangular WZW model has the following classical action

$$S^{qWZ}[\eta_L, \eta_R, \lambda_\mu] = S_L^{qWZ}[\eta_L(\tau)] + S_R^{qWZ}[\eta_R(\tau)] + \int d\tau \lambda_\mu(\tau) (a_L^\mu(\tau) + a_R^\mu(\tau)). \quad (1.26)$$

Here $\eta_L = (k_L, a_L^\mu)$, $\eta_R = (k_R, a_R^\mu)$ with $k_{L,R} \in LG_0$. The left and right chiral actions $S_L^{qWZ}(k_L, a_L^\mu)$ and $S_R^{qWZ}(k_R, a_R^\mu)$ have exactly the same dependence on their respective variables, but a_L^μ 's run over the positive standard Weyl alcove \mathcal{A}_+ and a_R^μ 's over the negative one \mathcal{A}_- . Finally, the fields λ_μ are the Lagrange multipliers. We note, that in the limit $q \rightarrow 1$ the action (1.26) reduces to the classical action of the standard full left-right WZW model written in the form of Ref.[13].

Remark: The variation of the action (1.24) does not depend on the choice of the symplectic potential θ_L^{qWZ} but only on the pair $(\omega_L^{qWZ}, H_L^{qWZ})$. This explains why one can give the meaning to the *classical* action (1.24) also in the case where ω_L^{qWZ} is not exact (i.e. there is no globally defined θ_L^{qWZ} such that (1.25) is valid).

6. Quasitriangular current algebra. It is of the crucial importance to understand the symmetries of the models (1.24) and (1.26). We have learned already from the standard ($q = 1$) chiral WZW example, that the canonical quantization of the model relies heavily on its symmetry structure. In fact, one needs the identification of suitable observables whose Poisson brackets get promoted to the quantum commutation relations. In the $q = 1$ case, such

observables are the components of the Kac-Moody current $j = \kappa \partial_\sigma m m^{-1}$ who serve as the Hamiltonian charges generating the action of the $G = LG_0$ on the phase space $M_L^{WZ} = LG_0 \times \mathcal{A}_+$. The latter fact can be expressed pregnantly by the following matrix Poisson brackets

$$\{k(\sigma) \circledast j(\sigma')\}_{WZ} = 2\pi C \delta(\sigma - \sigma') (k(\sigma) \otimes 1), \quad (1.27)$$

$$\{j(\sigma) \circledast j(\sigma')\}_{WZ} = \pi \delta(\sigma - \sigma') [C, j(\sigma) \otimes 1 - 1 \otimes j(\sigma')] + 2\pi \kappa C \partial_\sigma \delta(\sigma - \sigma'). \quad (1.28)$$

We note that the quantum versions of (1.27) and of (1.28) means respectively that k is the Kac-Moody primary field and that j generates the action of LG_0 on the quantum Hilbert space.

The present paper brings some clarification even of the symmetry structure of the standard chiral $q = 1$ WZW model. In fact, the second floor master model (1.2) is strictly symmetric with respect to the left \tilde{G} action. The two-step symplectic reduction down to the chiral WZW model reduces this exact \tilde{G} -symmetry to anomalous LG_0 -symmetry (1.28), generated by the current j .

It turns out that the quasitriangular picture is exactly analogous. The deformed chiral master model is strictly \tilde{G} Poisson-Lie symmetric with respect to the left action of \tilde{G} . This means that the chiral master Hamiltonian \tilde{H}_L is strictly \tilde{G} -invariant but \tilde{G} -invariance of the symplectic form Ω^q is broken in certain special (Poisson-Lie) way. Then also the \tilde{G} -invariance of the deformed chiral classical action (1.24) is broken in the special way dictated by the Poisson-Lie symmetry (see [4] for a nice general discussion of this issue). The two-step symplectic reduction then changes the strict \tilde{G} Poisson-Lie symmetry into anomalous LG_0 Poisson-Lie symmetry, whose non-Abelian moment map will satisfy the q -deformed version of the current algebra. The precise way how the central anomaly manifests itself follows from the fundamental Poisson bracket (1.6) defining the chiral symplectic form ω_L^{WZ} .

In order to find the quasitriangular analogue of the Kac-Moody current $j \in \mathcal{G} = LG_0$, we first write j in the variables (k, a^μ) :

$$j = -\kappa a^\mu k T^\mu k^{-1} + \kappa \partial_\sigma k k^{-1}. \quad (1.29)$$

It then follows

$$(j, \cdot)_{\mathcal{G}} = (\widetilde{Coad_k} \hat{\phi}_\kappa)', \quad (1.30)$$

where all necessary notations were already defined after Eq. (1.20). The q -Kac-Moody current $F(k, a^\mu) \in B$ is in turn given by

$$F = (\widetilde{Dres}_k e^{\hat{\Lambda}(\hat{\phi}_\kappa)})', \quad (1.31)$$

in full analogy with (1.30). The Poisson brackets involving F follow from the defining formula (1.6). Their calculation is somewhat involved but the result is very simple. It reads

$$\{k \otimes F\}_{qWZ} = \hat{r}^\kappa(k \otimes F), \quad (1.32)$$

$$\{F \otimes F\}_{qWZ} = [\hat{r}, F \otimes F], \quad (1.33)$$

$$\{F^\dagger \otimes F^\dagger\}_{qWZ} = -[\hat{r}, F^\dagger \otimes F^\dagger], \quad (1.34)$$

$$\{(F^\dagger)^{-1} \otimes F\}_{qWZ} = \hat{r}^{2\kappa}((F^\dagger)^{-1} \otimes F) - ((F^\dagger)^{-1} \otimes F)\hat{r}. \quad (1.35)$$

The r -matrices \hat{r}^κ and $\hat{r}^{2\kappa}$ are detailed in Section 5.2.5 and 5.2.6. The superscript κ over \hat{r}^κ indicates the presence of the central extension. In the limit $q \rightarrow 1$, the q -primary field condition (1.32) becomes (1.27) and the q -current algebra (1.33)-(1.35) yields (1.28).

Remarks: i) We observe that the quantity $\widetilde{Coad}_k \hat{\phi}_\kappa \in \tilde{\mathcal{G}}^*$ is *the* observable of the standard chiral WZW model. Its \mathcal{G}^* -part $(\widetilde{Coad}_k \hat{\phi}_\kappa)'$ gives the Kac-Moody current j , its central circle part $(\widetilde{Coad}_k \hat{\phi}_\kappa)^\infty$ gives the level κ and its σ -shift part $(\widetilde{Coad}_k \hat{\phi}_\kappa)^0$ contributes to the Hamiltonian H_L^{WZ} . The similar observation can be made also for the variable $\widetilde{Dres}_k e^{\hat{\Lambda}(\hat{\phi}_\kappa)} \in \tilde{B}$ in the quasitriangular case.

ii) It is worth stressing again that the q -Kac-Moody brackets (1.33)-(1.35) can be *derived* from the fundamental exchange relation (1.6) containing the elliptic dynamical r -matrix (1.7). This suggests, in particular, that there might be a lurking q -current algebra also in the description of the standard WZW conformal blocks on elliptic curves, Hitchin systems etc.

It turns out convenient to introduce a new dynamical variable defined by the relation

$$L = FF^\dagger. \quad (1.36)$$

The Poisson brackets (1.33) - (1.35) can be then equivalently rewritten in terms of only one relation:

$$\{L(\sigma) \otimes L(\sigma')\}_{qWZ} = (L(\sigma) \otimes L(\sigma'))\varepsilon\hat{r}(\sigma - \sigma') + \varepsilon\hat{r}(\sigma - \sigma')(L(\sigma) \otimes L(\sigma'))$$

$$-(1 \otimes L(\sigma')) \varepsilon \hat{r}(\sigma - \sigma' + 2i\varepsilon\kappa)(L(\sigma) \otimes 1) - (L(\sigma) \otimes 1) \varepsilon \hat{r}(\sigma - \sigma' - 2i\varepsilon\kappa)(1 \otimes L(\sigma')). \quad (1.37)$$

This relation coincides with the definition of q -current algebra introduced by Reshetikhin and Semenov-Tian-Shansky [42]. They call $L(\sigma)$ the q -current. By abuse of notation we shall refer to both quantities F and L as to the q -currents.

Recall that the non-deformed current j can be simply expressed in terms of the primary field $m(\sigma)$:

$$j = \kappa \partial_\sigma m m^{-1}. \quad (1.38)$$

This relation can be called the classical Knizhnik-Zamolodchikov equation [37] since its quantum version becomes indeed the KZ-equation written in the operatorial form [24]. It turns out, that the q -current $L(\sigma)$ can be also simply expressed in terms of the q -primary field $m(\sigma)$:

$$L(\sigma) = m(\sigma + i\kappa\varepsilon)m^{-1}(\sigma - i\varepsilon\kappa). \quad (1.39)$$

This nice relation can be interpreted as the classical q -KZ equation. As expected, it is not differential but rather a difference equation.

7. The plan of the paper: in Chapter 2 we first explain the crucial notion of the central biextension \tilde{G} of a Lie group G and we derive explicit formulae for adjoint and coadjoint actions of \tilde{G} . Then we detail our basic observation that the two-step symplectic reduction of the master model (1.1) on \tilde{G} gives the standard WZW model on G . We shall also see that this construction can be performed for any central biextension; the affine Kac-Moody group being only the special case. We are thus led to the notion of the universal WZW model.

In Chapter 3, we study the case of the affine Kac-Moody group. We show that the master model on \tilde{G} can be decomposed in two copies of the simpler chiral model. Then we perform the two-step chiral symplectic reduction to obtain the standard chiral WZW model and we detail the symplectic structure of the model in the (k, a^μ) variables.

We devote Chapter 4 to the construction of the universal quasitriangular model based on any Drinfeld double $\tilde{\tilde{D}}$ of the biextended group \tilde{G} . We first review some basic notions of the theory of the Poisson-Lie groups and then we identify which conditions $\tilde{\tilde{D}}$ must fulfil in order that the two-step reduction could be performed. We call such good doubles $\tilde{\tilde{D}}$ the WZW

doubles of \tilde{G} . The rest of the chapter is devoted to the construction of the affine Lu-Weinstein-Soibelman double \tilde{G}^C and to the proving that it does fulfil the required conditions. The quasitriangular WZW model based on this particular double is the q -deformation of the standard loop group WZW model.

The core of the paper is Chapter 5. Starting from the affine Lu-Weinstein-Soibelman double, we first explain the construction of the deformed chiral geodesical model (1.3). Then we perform the two step symplectic reduction down to the quasitriangular chiral WZW model. We shall make explicit the symplectic structure ω_L^{qWZ} and the Hamiltonian H_L^{qWZ} of the model, we introduce the q -current algebra and show how its commutation relations follows from the symplectic structure ω_L^{qWZ} . We also show that the model has the correct $q \rightarrow 1$ limit and finally we glue up the two quasitriangular chiral WZW models to obtain the full left-right theory.

In Chapter 6 we summarize the results and provide an outlook. In particular, we outline the further generalizations of the construction aiming to the q -deformation of the whole WZW factory, we draw the plausible picture of the quantization of the model and we furnish also some remarks on the role of the Virasoro group in the q -deformed case.

Chapter 7 contains Appendices that are of three types : i) they provide more background material for better understanding of the article; ii) they contain the detailed technical proofs of some assertions in the text; iii) they give alternative derivations of some results.

Chapter 2

Universal WZW model

In this chapter, we shall be very general and we shall work with an arbitrary Lie group \hat{G} which is the central biextension of a Lie group G . The loop group case leading to the standard WZW model will be only the special (though very important) example of our construction. Indeed, we shall see that the WZW-like symplectic structure is "universal"; it can be defined not only for the loop groups and it does not depend on the detailed structure of the group multiplication or of the central extension.

2.1 Central biextension

Consider a central extension \hat{G} of a Lie group G by the circle group $U(1)$. This means that there is the exact sequence of morphisms of groups:

$$1 \rightarrow U(1) \rightarrow \hat{G} \xrightarrow{\pi} G \rightarrow 1; \quad (2.1)$$

where $U(1)$ is injected into the centre of \hat{G} . The morphism from \hat{G} to G is denoted as π . Note that the circle fibration over the base G can be topologically nontrivial. The fundamental example of this exact sequence is the famous central extension of the loop groups LG_0 . It is reviewed in the appendix 7.1. The reader is invited to consult this appendix whenever he (or she) will need the illustration of the general scheme presented in this chapter.

It is clear that the exact sequence (2.1) of groups induces the following exact sequence of their Lie algebras:

$$0 \rightarrow \mathbf{R} \rightarrow \hat{\mathcal{G}} \xrightarrow{\pi_*} \mathcal{G} \rightarrow 0. \quad (2.2)$$

Here $\pi_* : \hat{\mathcal{G}} \rightarrow \mathcal{G}$ is the Lie algebra homomorphism induced by the group homomorphism π . In general there need not exist a canonical map between $\hat{\mathcal{G}}$ and \mathcal{G} that would go in opposite direction than π_* . Suppose, however, that we choose ¹ such a map $\iota : \mathcal{G} \rightarrow \hat{\mathcal{G}}$. We do not suppose, however, that ι is the homomorphism of Lie algebras! We just claim that ι be a linear injection of \mathcal{G} into $\hat{\mathcal{G}}$ fulfilling the following condition:

$$\pi_*(\iota(\xi)) = \xi \quad (2.3)$$

for every $\xi \in \mathcal{G}$.

The existence of the map ι immediately implies, that the structure of the Lie algebra $\hat{\mathcal{G}}$ of \hat{G} must be given by the following commutator²

$$[\hat{\xi}, \hat{\eta}] = \iota([\pi_*\hat{\xi}, \pi_*\hat{\eta}]) + \rho(\pi_*\hat{\xi}, \pi_*\hat{\eta})\hat{T}^\infty \quad (2.4)$$

for some cocycle $\rho : \mathcal{G} \wedge \mathcal{G} \rightarrow \mathbf{R}$. Recall that the cocycle condition means

$$\rho([\xi, \eta], \zeta) + \rho([\eta, \zeta], \xi) + \rho([\zeta, \xi], \eta) = 0, \quad \xi, \eta, \zeta \in \mathcal{G}. \quad (2.5)$$

The elements $\hat{\xi}, \hat{\eta}$ in (2.4) are from $\hat{\mathcal{G}}$, the element $\hat{T}^\infty \in \hat{\mathcal{G}}$ corresponds to the generator of $U(1)$ injected in \hat{G} according to the exact sequence (2.1). In other words:

$$\pi_*\hat{T}^\infty = 0. \quad (2.6)$$

In particular, the relation (2.4) implies

$$[\iota(\xi), \iota(\eta)] = \iota([\xi, \eta]) + \rho(\xi, \eta)\hat{T}^\infty. \quad (2.7)$$

Suppose that there is a one-parameter subgroup \hat{S} of automorphisms of \hat{G} commuting with the central circle action. It then gives rise to the one-parameter subgroup S of automorphisms of G . We denote as ∂ the generator of S and we define

¹If \hat{G} is constructed from G in a suitable way, the existence of a natural ι may be the consequence of this construction; this happens in the case of the central extensions of the loop groups of simple compact Lie groups.

²We use the same symbol $[\cdot, \cdot]$ for the commutators of different Lie algebras. It should be clear which usage we have in mind by realizing to which Lie algebra the arguments of the commutator belong.

Definition 2.1: The group $\tilde{G} = \mathbf{R} \times_{\hat{S}} \hat{G}$ is the central biextension if

$$\rho(\xi, \eta) = (\xi, \partial\eta)_{\mathcal{G}}, \quad (2.8)$$

where $(\cdot, \cdot)_{\mathcal{G}}$ is a symmetric non-degenerate invariant bilinear form on $\mathcal{G} = Lie(G)$, such that

$$(\xi, \partial\eta)_{\mathcal{G}} + (\partial\xi, \eta)_{\mathcal{G}} = 0. \quad (2.9)$$

If \tilde{G} is the central biextension then there is the canonical symmetric non-degenerate invariant bilinear form on $\tilde{\mathcal{G}} = Lie(\tilde{G})$. It is given by

$$((iX, \xi, ix), (iY, \eta, iy))_{\tilde{\mathcal{G}}} = (\xi, \eta)_{\mathcal{G}} - Xy - Yx. \quad (2.10)$$

Conventions 2.2: The generator of \hat{S} in $\tilde{\mathcal{G}}$ will be denoted either as \tilde{T}^0 or as $(i, 0, 0)$. The elements $\iota(\xi)$ and \hat{T}^∞ of $\hat{\mathcal{G}}$ will be denoted either as $\tilde{\iota}(\xi)$ and \tilde{T}^∞ or as $(0, \xi, 0)$ and $(0, 0, i)$ when considered as the elements of the Lie algebra $\tilde{\mathcal{G}}$.

Remark: The theory of the biextension was developed by Medina and Revoy [33] in their quest of classifying the Lie algebras admitting the symmetric non-degenerate invariant bilinear form. They used the terminology "double extension". We take the liberty of modifying this name to the "biextension" since in our paper the word double will often appear in the different sense. It should be also noted that the extensions considered here form only the subclass of the Medina-Revoy extensions. For the physicists oriented survey of the subject see also [23].

It is not difficult to verify directly the invariance of the bilinear form $(\cdot, \cdot)_{\tilde{\mathcal{G}}}$. For this, it is also useful to write the commutator in $\tilde{\mathcal{G}}$ in terms of the notation above:

$$[(iX, \xi, ix), (iY, \eta, iy)] = (i0, [\xi, \eta] + X\partial\eta - Y\partial\xi, i(\xi, \partial\eta)_{\mathcal{G}}). \quad (2.11)$$

The following theorem is of great importance for our construction:

Theorem 2.3: Suppose that an element $\hat{g} \in \hat{G}$ is viewed as the element of \tilde{G} , where \tilde{G} is the central biextension of G . Then the adjoint action of \hat{g} on the Lie algebra $\tilde{\mathcal{G}}$ has the following explicit form

$$\widetilde{Ad}_{\hat{g}}(iX, \xi, ix) = (iX, Ad_g\xi - X\partial gg^{-1}, ix - i(g^{-1}\partial g, \xi)_{\mathcal{G}} + \frac{1}{2}iX(g^{-1}\partial g, g^{-1}\partial g)_{\mathcal{G}}). \quad (2.12)$$

Conventions 2.4: Since the large part of our technical work will consist in "travelling" between the groups G, \hat{G} and \tilde{G} , we should be very careful in book-keeping with respect to which group the certain operations are considered. Thus, e.g. \widetilde{Ad} means that the adjoint operation is taken with respect to the group \tilde{G} and we shall always use the convention that $g = \pi(\hat{g})$, if both g and \hat{g} appear in the same formula. Moreover we denote

$$-\partial gg^{-1} \equiv Ad_g \partial - \partial; \quad g^{-1} \partial g \equiv Ad_{g^{-1}} \partial - \partial. \quad (2.13)$$

The Ad operation in (2.13) is taken in the group $\mathbf{R} \times_S G$. We do not denote this group by a special symbol because it will appear less frequently than its colleagues mentioned above. Finally, it is clear that both ∂gg^{-1} and $g^{-1} \partial g$ live in \mathcal{G} .

Remark: In the context of the biextensions of the loop groups, ∂gg^{-1} equals to $\partial_\sigma gg^{-1}$, where ∂_σ is the derivative with respect to the loop parameter.

Proof of the theorem 2.3: First we show that

$$\widetilde{Ad}_{\hat{g}}(0, \xi, ix) = (0, Ad_g \gamma, ix - i(g^{-1} \partial g, \xi)_{\mathcal{G}}). \quad (2.14)$$

In this special case, we can forget about the group \tilde{G} and can work only with \hat{G} . The reason is that $(0, \xi, x)$ is in $\hat{\mathcal{G}}$ and \hat{G} is the subgroup of \tilde{G} . First of all we have

$$\pi_*(\widehat{Ad}_{\hat{g}} \iota(\xi)) = Ad_g \xi, \quad (2.15)$$

because π is the homomorphism of groups. Moreover,

$$\widehat{Ad}_{\hat{g}} \hat{T}^\infty = \hat{T}^\infty, \quad (2.16)$$

because \hat{T}^∞ is the generator of the central circle. Thus we conclude, that

$$\widehat{Ad}_{\hat{g}}(\iota(\xi) + x \hat{T}^\infty) = \iota(Ad_g \xi) + (x - F(g, \xi)) \hat{T}^\infty, \quad (2.17)$$

where $F(g, \xi)$ is a function to be determined. First of all, we know $F(g, \xi)$ at the group origin $g = e$ (where it vanishes) and near the group origin

$$F(\chi, \xi) = (\partial \chi, \xi)_{\mathcal{G}}, \quad \chi \in \mathcal{G}. \quad (2.18)$$

Moreover, it is easy to check that $F(g, \xi)$ has to verify the following cocycle condition

$$F(g_1 g_2, \xi) = F(g_2, \xi) + F(g_1, Ad_{g_2} \xi). \quad (2.19)$$

Infinitesimally:

$$F(g_1\chi, \xi) = (\partial\chi, \xi)_{\mathcal{G}} + F(g_1, [\chi, \xi]). \quad (2.20)$$

Thus we obtain a first order differential equation with the known initial condition. One readily checks that

$$F(g, \xi) = (g^{-1}\partial g, \xi)_{\mathcal{G}} \quad (2.21)$$

is its solution.

The proof of the formula (2.12) with $X \neq 0$ then goes like in [41]; i.e; we know that the r.h.s. of (2.12) must have the form (X, \dots, \dots) and then one directly checks that the formula (2.12) is the only possible one preserving the invariance of the bilinear form $(\cdot, \cdot)_{\tilde{\mathcal{G}}}$.

The theorem is proved. #

It will be also useful to have the explicit expressions for the invariant bilinear form on the dual $\tilde{\mathcal{G}}^*$ and for the coadjoint action of \hat{g} on $\tilde{\mathcal{G}}^*$. They read, respectively

$$((A, \alpha, a)^*, (B, \beta, b)^*)_{\tilde{\mathcal{G}}^*} = (\alpha, \beta)_{\mathcal{G}^*} - Ab - Ba; \quad (2.22)$$

$$\begin{aligned} & \widetilde{Coad}_{\hat{g}}(C, \gamma, c)^* = \\ & = (C + \langle \gamma, g^{-1}\partial g \rangle + \frac{1}{2}c(g^{-1}\partial g, g^{-1}\partial g)_{\mathcal{G}}, Coad_g\gamma + c\Upsilon^{-1}(\partial g g^{-1}), c)^*. \end{aligned} \quad (2.23)$$

We have also

$$\widehat{Coad}_{\hat{g}}(\pi^*(\gamma) + c\hat{t}_{\infty}) = \pi^*(Coad_g\gamma + c\Upsilon^{-1}(\partial g g^{-1})) + c\hat{t}_{\infty}. \quad (2.24)$$

Conventions 2.5 : The map $\Upsilon : \mathcal{G}^* \rightarrow \mathcal{G}$ is defined by the invariant bilinear form $(\cdot, \cdot)_{\mathcal{G}}$. The decomposition $\tilde{\mathcal{G}} = \mathbf{R}\tilde{T}^0 + \tilde{\mathcal{I}}(\mathcal{G}) + \mathbf{R}\tilde{T}^{\infty}$ induces the decomposition of the dual $\tilde{\mathcal{G}}^*$, hence every $\tilde{\alpha} \in \tilde{\mathcal{G}}^*$ can be cast as³

$$\tilde{\alpha} = (A, \alpha, a)^*, \quad (2.25)$$

where A, a are in \mathbf{R} and α in \mathcal{G}^* . The element $(1, 0, 0)^*$ will be sometimes denoted as \tilde{t}_0 , $(0, \alpha, 0)^*$ as $\tilde{\pi}^*(\alpha)$ and $(0, 0, 1)^*$ as \tilde{t}_{∞} .

³Note that there is no i in the dual objects. It is consistent to use such a notation because we shall never use in this paper the dual of the *complexified* algebra $\tilde{\mathcal{G}}^{\mathbf{C}}$.

In a similar way the decomposition $\hat{\mathcal{G}} = \iota(\mathcal{G}) + \mathbf{R}\hat{T}^\infty$ induces

$$\hat{\alpha} = \pi^*(\alpha) + a\hat{t}_\infty, \quad \hat{\alpha} \in \hat{\mathcal{G}}^*, \quad \alpha \in \mathcal{G}^*, \quad (2.26)$$

where $\hat{t}_\infty \in \hat{\mathcal{G}}^*$ is characterized by the property

$$\langle \hat{t}_\infty, \iota(\mathcal{G}) \rangle = 0, \quad \langle \hat{t}_\infty, \hat{T}^\infty \rangle = 1. \quad (2.27)$$

Of course, $\pi^* : \mathcal{G}^* \rightarrow \hat{\mathcal{G}}^*$ is the map dual to $\pi_* : \hat{\mathcal{G}} \rightarrow \mathcal{G}$ defined by the exact sequence (2.2). Note also that by definition

$$\langle \widetilde{Coad}_{\hat{g}} \tilde{\gamma}, \tilde{\xi} \rangle = \langle \tilde{\gamma}, \widetilde{Ad}_{\hat{g}^{-1}} \tilde{\xi} \rangle, \quad (2.28)$$

where the pairing $\langle \cdot, \cdot \rangle$ is clearly defined as

$$\langle (C, \gamma, c)^*, (X, \xi, x) \rangle = CX + cx + \langle \gamma, \xi \rangle. \quad (2.29)$$

2.2 The symplectic reduction

2.2.1 Second floor master model on \tilde{G} .

As we have stated in the introduction, the classical action of the geodesical model on \tilde{G} reads

$$\tilde{S}(\tilde{g}) = -\frac{\kappa}{4} \int d\tau (\tilde{g}^{-1} \frac{d}{d\tau} \tilde{g}, \tilde{g}^{-1} \frac{d}{d\tau} \tilde{g})_{\tilde{g}}, \quad (2.30)$$

where $\tilde{g}(\tau) \in \tilde{G}$ and κ is a parameter going to play the role of the level of the WZW model. We first rewrite this action in the first order Hamiltonian form:

$$\tilde{S}(\tilde{\beta}_L, \tilde{g}) = \int d\tau [\langle \tilde{\beta}_L, \frac{d}{d\tau} \tilde{g} \tilde{g}^{-1} \rangle + \frac{1}{\kappa} (\tilde{\beta}_L, \tilde{\beta}_L)_{\tilde{g}^*}]. \quad (2.31)$$

Here $\tilde{\beta}_L \in \tilde{\mathcal{G}}^*$ are the "momentum" and \tilde{g} the position coordinates induced by the right trivialization of the phase space $T^*\tilde{G}$. The symplectic potential on $T^*\tilde{G}$ is

$$\tilde{\theta} = \langle \tilde{\beta}_L, d\tilde{g} \tilde{g}^{-1} \rangle \quad (2.32)$$

in agreement with the general story explained after the formula (1.3) in the Introduction. The reader can find in Appendix 7.2 the detailed account of the canonical symplectic structure on the cotangent bundle of the group manifold. It is clear that we can obtain (2.30) from (2.31) by eliminating $\tilde{\beta}_L$ via the field equations.

2.2.2 The symplectic reduction to the first floor \hat{G}

The field multiplet $\tilde{\beta}_L \in \tilde{\mathcal{G}}^*, \tilde{g} \in \tilde{G}$ of (2.31) can be also parametrized as follows

$$\tilde{\beta}_L = \widetilde{Coad}_u \tilde{\gamma}_L, \quad \tilde{g} = u \hat{g} u, \quad (2.33)$$

where $\tilde{\gamma}_L \in \tilde{\mathcal{G}}^*$, $u = \exp s \tilde{T}^0 \in \tilde{G}$ and $\hat{g} \in \hat{G}$. Using the invariance of the form $(\cdot, \cdot)_{\tilde{\mathcal{G}}^*}$, the action written in the new variables becomes

$$\tilde{S}(\tilde{\gamma}_L, \hat{g}, s) = \int d\tau [\langle \tilde{\gamma}_L, (\tilde{T}^0 + \hat{g} \tilde{T}^0 \hat{g}^{-1}) \rangle \frac{ds}{d\tau} + \langle \tilde{\gamma}_L, \frac{d}{d\tau} \hat{g} \hat{g}^{-1} \rangle + \frac{1}{\kappa} (\tilde{\gamma}_L, \tilde{\gamma}_L)_{\tilde{\mathcal{G}}^*}]. \quad (2.34)$$

Following our list of conventions, we can represent $\tilde{\gamma}_L$ as

$$\tilde{\gamma}_L = (\gamma_L^0, \gamma_L, \gamma_L^\infty)^*. \quad (2.35)$$

We can introduce once again a new way of parametrizing the phase space $T^* \tilde{G}$, now by the set of coordinates $(\gamma_L^s, \gamma_L, \gamma_L^\infty, s, \hat{g})$ where

$$\gamma_L^s = \langle \tilde{\gamma}_L, (\tilde{T}^0 + \hat{g} \tilde{T}^0 \hat{g}^{-1}) \rangle = 2\gamma_L^0 - \langle \gamma_L, \partial g g^{-1} \rangle + \gamma_L^\infty \frac{1}{2} (g^{-1} \partial g, g^{-1} \partial g)_G. \quad (2.36)$$

The action in these newest coordinates becomes

$$\begin{aligned} \tilde{S}(\gamma_L^s, \gamma_L, \gamma_L^\infty, \hat{g}, s) = & \int d\tau [\gamma_L^s \frac{ds}{d\tau} - \frac{1}{\kappa} \gamma_L^s \gamma_L^\infty] \\ & + \int d\tau [\langle \hat{\gamma}_L, \frac{d}{d\tau} \hat{g} \hat{g}^{-1} \rangle + \frac{1}{\kappa} (\gamma_L, \gamma_L)_G - \frac{\gamma_L^\infty}{\kappa} \langle \gamma_L, \partial g g^{-1} \rangle + \frac{1}{2\kappa} (\gamma_L^\infty)^2 (g^{-1} \partial g, g^{-1} \partial g)_G]. \end{aligned} \quad (2.37)$$

Here we have used the formula (2.12), the explicit form (2.22) of the scalar product on $\tilde{\mathcal{G}}^*$, and we have set

$$\hat{\gamma}_L = (0, \gamma_L, \gamma_L^\infty)^*. \quad (2.38)$$

The reader should pay attention to the distribution of hats and tildes. Of course, we still use the convention that $g = \pi(\hat{g})$, if both g and \hat{g} appear in the same formula.

The terms containing (not containing) the level κ encodes the Hamiltonian (the symplectic potential $\tilde{\theta}$) in our new coordinates. It is clear that the

coordinate γ_L^s Poisson-commute⁴ with the Hamiltonian since the latter does not contain the variable s . We can therefore consistently set $\tilde{\gamma}_L^s = 0$ in the action (2.37). This is the so-called symplectic reduction of the dynamical system (2.31) with respect to the moment map generating the axial action of \tilde{T}^0 on $T^*\tilde{G}$. The resulting reduced action reads

$$\begin{aligned} \hat{S}(\hat{\gamma}_L, \hat{g}) = \\ \int d\tau [\langle \hat{\gamma}_L, \frac{d}{d\tau} \hat{g} \hat{g}^{-1} \rangle + \frac{1}{\kappa} (\gamma_L, \gamma_L)_{\mathcal{G}^*} - \frac{\gamma_L^\infty}{\kappa} \langle \gamma_L, \partial g g^{-1} \rangle + \frac{1}{2\kappa} (\gamma_L^\infty)^2 (g^{-1} \partial g, g^{-1} \partial g)_{\mathcal{G}}]. \end{aligned} \quad (2.39)$$

It can be rewritten entirely from the point of view of the cotangent bundle $T^*\hat{G}$. Indeed, we shall first note that the symplectic potential $\hat{\theta}$ of the reduced dynamical system (2.39) is

$$\hat{\theta} = \langle \hat{\gamma}_L, d\hat{g} \hat{g}^{-1} \rangle \quad (2.40)$$

which is the canonical symplectic potential on $T^*\hat{G}$ written in the right trivialization coordinates $\hat{\gamma}_L, \hat{g}$. The Hamiltonian \hat{H} of the reduced model can be also written elegantly in terms of natural quantities related to $T^*\hat{G}$. Indeed, it turns out that

$$\hat{H} = -\frac{1}{2\kappa} (\iota^*(\hat{\gamma}_L), \iota^*(\hat{\gamma}_L))_{\mathcal{G}^*} - \frac{1}{2\kappa} (\iota^*(\hat{\gamma}_R), \iota^*(\hat{\gamma}_R))_{\mathcal{G}^*}, \quad (2.41)$$

where $\iota^* : \hat{\mathcal{G}}^* \rightarrow \mathcal{G}^*$ is the map dual to the injection $\iota : \mathcal{G} \rightarrow \hat{\mathcal{G}}$ and $\hat{\gamma}_R$ is the coordinate on $T^*\hat{G}$ given by the left trivialization. Note also the minus signs reflecting the fact that the form $(\cdot, \cdot)_{\mathcal{G}}$ is negative definite. In other words, $\hat{\gamma}_L \hat{g} = \hat{g} \hat{\gamma}_R$ or $\hat{\gamma}_L = \widehat{Coad_{\hat{g}}} \hat{\gamma}_R$. In deriving (2.41), we have used the formula (2.24) expressing the coadjoint action on $\hat{\mathcal{G}}^*$.

We conclude that the reduced first floor action \hat{S} can be written in a completely left-right symmetric way as

$$\begin{aligned} \hat{S} = \frac{1}{2} \int d\tau [\langle \hat{\gamma}_L, \frac{d}{d\tau} \hat{g} \hat{g}^{-1} \rangle + \langle \hat{\gamma}_R, \hat{g}^{-1} \frac{d}{d\tau} \hat{g} \rangle] \\ + \frac{1}{2\kappa} \int d\tau [(\iota^*(\hat{\gamma}_L), \iota^*(\hat{\gamma}_L))_{\mathcal{G}^*} + (\iota^*(\hat{\gamma}_R), \iota^*(\hat{\gamma}_R))_{\mathcal{G}^*}]. \end{aligned} \quad (2.42)$$

⁴Of course, this can be seen also from the fact that γ_L^s is the moment map generating the axial action $\tilde{g} \rightarrow u \tilde{g} u$, $\tilde{\gamma}_L \rightarrow Coad_u \tilde{\gamma}_L$, $u = \exp s \tilde{T}^0$ and the Hamiltonian $\tilde{H} = -\frac{1}{2\kappa} (\tilde{\gamma}_L, \tilde{\gamma}_L)_{\tilde{\mathcal{G}}^*}$ is clearly invariant with respect to this action.

Remark: The first floor universal WZW action (2.42) can be naturally written for whatever central extension of the group G admitting the biinvariant metric. In other words, even if the cocycle $\rho(x, y)$ does not have the form (2.8), the action (2.42) makes sense. Actually, we had begun to write this article from the vantage point of the action (2.42). This seemed sufficient for our deformation programme since the symplectic structure of the model on $T^*\hat{G}$ is already canonical. Thus we can introduce the Heisenberg double, Semenov-Tian-Shansky form etc. What happened was, however, that we had canonically obtained the symplectic structure of the deformed model but there was no clue how to single out the canonical choice of the deformed Hamiltonian. In fact, it turned out that many Hamiltonians have satisfied the basic condition of the Poisson-Lie symmetry and have had correct limit when the deformation parameter went to zero. It was the search of the natural Hamiltonian that finally opened our eyes and we realized that the model (2.42) can be lifted to the second floor \tilde{G} for the cocycles of the type (2.8). On \tilde{G} , there is the canonical choice of the Hamiltonian even in the deformed case. Then the canonical Hamiltonian on the first floor is the one inherited from the master model on \tilde{G} .

2.2.3 The reduction to the ground floor G

The second symplectic reduction is slightly more involved since the central circle bundle over \hat{G} is nontrivial. In the Appendix 7.3 the reduction is performed by working directly with the Poisson brackets on $T^*\hat{G}$ and T^*G . Here we shall rather work with the symplectic forms. This form language has the advantage of being briefer and for this reason we expose it in the main body of the paper. However, the dual (Poisson bracket) derivation has the advantage of being more transparently deformable to the case of the nontrivial Drinfeld doubles. Anyway, we offer both derivations in this paper.

Recall that the canonical symplectic form $\hat{\omega}$ on $T^*\hat{G}$ is given by $\hat{\omega} = d\hat{\theta}$ where $\hat{\theta}$ is the symplectic potential:

$$\hat{\theta} = +\langle \hat{\gamma}_L, d\hat{g}\hat{g}^{-1} \rangle. \quad (2.43)$$

In the same way, θ is the symplectic potential on T^*G :

$$\theta = +\langle \gamma_L, dg g^{-1} \rangle. \quad (2.44)$$

Of course, we have trivialized the cotangent bundle T^*G by the right-invariant forms hence every point $K \in T^*G$ can be decomposed as $K = \gamma_L g$, where

$\gamma_L \in \mathcal{G}^*$ and $g \in G$. In this section 2.2.3 we shall never work with the opposite decomposition $K = g\gamma_R$.

Consider an extension $\pi_{ext} : T^*\hat{G} \rightarrow T^*G$ of the map $\pi : \hat{G} \rightarrow G$ defined by the exact sequence (2.1). π_{ext} can be easily expressed in the right trivialization as follows

$$\pi_{ext}(\hat{\gamma}_L \hat{g}) = \iota^*(\hat{\gamma}_L) \pi(\hat{g}). \quad (2.45)$$

Recall that ι^* is the map dual to $\iota : \mathcal{G} \rightarrow \hat{\mathcal{G}}$. Then there is a natural relation between the forms $\hat{\theta}$ and θ as the following lemma states:

Lemma 2.6: It holds:

$$\hat{\theta} = \gamma_L^\infty(R_{\hat{g}^{-1}}^* \hat{t}_\infty) + \pi_{ext}^* \theta. \quad (2.46)$$

Proof: Let T^i be a basis of the Lie algebra \mathcal{G} and t_i its dual basis in \mathcal{G}^* . We can then choose $\hat{T}^\infty, \iota(T^i)$ as the basis in $\hat{\mathcal{G}}$. Its dual is clearly $\hat{t}_\infty, \pi^*(t_i)$, since $\pi_* \circ \iota$ equals to the identity map $\mathcal{G} \rightarrow \mathcal{G}$ (cf. (2.3)). The right invariant Maurer-Cartan form $\rho_{\hat{G}} = d\hat{g}\hat{g}^{-1}$ can be clearly written as

$$d\hat{g}\hat{g}^{-1} = R_{\hat{g}^{-1}}^* \hat{t}_\infty \otimes \hat{T}^\infty + R_{\hat{g}^{-1}}^* \pi^*(t_i) \otimes \iota(T^i), \quad (2.47)$$

where R^* is the pull-back map. Then the symplectic potential $\hat{\theta}$ can be expressed as follows

$$\hat{\theta} = \gamma_L^\infty(R_{\hat{g}^{-1}}^* \hat{t}_\infty) + \langle \hat{\gamma}_L, \iota(T^i) \rangle (R_{\hat{g}^{-1}}^* \pi^*(t_i)). \quad (2.48)$$

Now we observe that:

1) $\langle \gamma_L, T^i \rangle$ is a function on T^*G . If we calculate its π_{ext} -pullback we obtain

$$\pi_{ext}^* \langle \gamma_L, T^i \rangle = \langle \iota^*(\hat{\gamma}_L), T^i \rangle = \langle \hat{\gamma}_L, \iota(T^i) \rangle. \quad (2.49)$$

2) Due to the fact that $\pi : \hat{G} \rightarrow G$ is the group homomorphism, we have

$$R_{\hat{g}^{-1}}^* \pi^* = \pi^* R_{\pi(\hat{g})^{-1}}^*. \quad (2.50)$$

Using 1) and 2) in (2.48) we infer that

$$\hat{\theta} = \gamma_L^\infty(R_{\hat{g}^{-1}}^* \hat{t}_\infty) + \pi_{ext}^* \langle \gamma_L, T^i \rangle \pi^* R_{\pi(\hat{g})^{-1}}^* t_i = \gamma_L^\infty(R_{\hat{g}^{-1}}^* \hat{t}_\infty) + \pi_{ext}^* \theta. \quad (2.51)$$

The lemma is proved. #

By using the formula

$$d(d\hat{g}\hat{g}^{-1}) = d\hat{g}\hat{g}^{-1} \wedge d\hat{g}\hat{g}^{-1} \quad (2.52)$$

for the exterior derivative of the Maurer-Cartan form, we can immediately calculate also the exterior derivatives of its components. Thus we obtain, in particular, that

$$d(R_{\hat{g}^{-1}}^* \hat{t}_\infty) = \frac{1}{2} \rho(T^i, T^j) (R_{\hat{g}^{-1}}^* \pi^*(t_i)) \wedge (R_{\hat{g}^{-1}}^* \pi^*(t_j)) = \frac{1}{2} \pi_{ext}^* \rho(dgg^{-1} \frown dgg^{-1}), \quad (2.53)$$

where $\rho(.,.)$ is the cocycle defining the central extension. We can therefore express conveniently the symplectic form $\hat{\omega}$ on $T^*\hat{G}$ as

$$\hat{\omega} = d\hat{\theta} = +d\gamma_L^\infty \wedge (R_{\hat{g}^{-1}}^* \hat{t}_\infty) + \frac{1}{2} \gamma_L^\infty \pi_{ext}^* \rho(dgg^{-1} \frown dgg^{-1}) + \pi_{ext}^* d\theta. \quad (2.54)$$

Now we can directly perform the symplectic reduction by setting

$$\gamma_L^\infty = \kappa. \quad (2.55)$$

The restriction of the form $\hat{\omega}$ to the submanifold determined by (2.55) is clearly a π_{ext}^* -pullback of the two-form ω_{red} living on the manifold T^*G and given by

$$\omega_{red} = \frac{\kappa}{2} \rho(dgg^{-1} \frown dgg^{-1}) + d\langle \gamma_L, dgg^{-1} \rangle. \quad (2.56)$$

The form ω_{red} is the reduced symplectic form as the notation indicates.

Theorem 2.7: If G is the loop group LG_0 then the form ω_{red} on T^*G is the symplectic form of the standard WZW model.

Proof: The loop group cocycle $\rho(\eta, \xi)$ reads

$$\rho(\eta, \xi) = \frac{1}{2\pi} \int_{S^1} (\eta, \partial_\sigma \xi)_{\mathcal{G}_0^C}. \quad (2.57)$$

The form ω_{red} can be then rewritten as

$$\omega_{red} = \frac{\kappa}{4\pi} \int_{S^1} (dgg^{-1} \frown \partial_\sigma(dgg^{-1}))_{\mathcal{G}_0} + \frac{1}{2\pi} d \int_{S^1} (J_L(\sigma), dgg^{-1})_{\mathcal{G}_0}. \quad (2.58)$$

Here $J_L(\sigma) = \Upsilon(\gamma_L)$, where Υ is the identification map $\mathcal{G}^* \rightarrow \mathcal{G}$ induced by $(\cdot, \cdot)_{\mathcal{G}} = \frac{1}{2\pi} \int (\cdot, \cdot)_{\mathcal{G}_0}$. It turns out that (2.58) is the standard WZW symplectic form of the reference [6]. (Actually, there is the difference in the overall normalization factor (-2π) ; if wished, this factor can be easily restored in all our formulae. The reader should also note that our bilinear form $(\cdot, \cdot)_{\mathcal{G}_0}$ is $-Tr$ of Ref.[6]).

The theorem is proved. #

The Hamiltonian \hat{H} of the first floor model on \hat{G} can be read off from the formula (2.41). It clearly Poisson-commutes with the moment map $\tilde{\gamma}_L^\infty$ since it is invariant with respect to the central circle action. It thus descends to the function on the ground-floor phase space T^*G where it is given by the formula

$$\begin{aligned} H_{WZW}(\gamma_L, g) &= -\frac{1}{\kappa}(\gamma_L, \gamma_L)_{\mathcal{G}^*} + \langle \gamma_L, \partial g g^{-1} \rangle - \frac{\kappa}{2}(g^{-1} \partial g, g^{-1} \partial g)_{\mathcal{G}} = \\ &= -\frac{1}{2\kappa}(J_L, J_L)_{\mathcal{G}} - \frac{1}{2\kappa}(J_R, J_R)_{\mathcal{G}}, \end{aligned} \quad (2.59)$$

where

$$J_L = \Upsilon(\gamma_L), \quad J_R(\sigma) = -Ad_{g^{-1}} J_L(\sigma) + \kappa g^{-1} \partial_\sigma g. \quad (2.60)$$

We immediately observe that our Hamiltonian H_{WZW} coincides (up the factor (2π) mentioned above) with the standard WZW Hamiltonian of Ref. [6]. Note that the symplectic form of Ref. [6] is the (-2π) -multiple of our ω_{red} and the Hamiltonian [6] is (2π) -multiple of our H_{WZW} . The discrepancy in the relative sign is innocent. Indeed, if we change the sign of the symplectic potential in (2.31) and integrate away the momenta, we shall again obtain the same second order action (2.30). Thus we have proved the following theorem

Theorem 2.8: The two step symplectic reduction of the master model (1.1) induced by equating $\gamma_L^s = 0$, $\gamma_L^\infty = \kappa$ yields the standard WZW model.

Remark: We stress that the dynamics of the WZW model is intrinsically left-right symmetric. The left-right asymmetry in the Hamiltonian (cf. (2.60)) is purely coordinate effect which can be traced back to the asymmetric way of performing the symplectic reduction. Indeed, the choice of the right trivialization of the bundle in (2.32) already breaks the symmetry. The left-right symmetric formalism

of Appendix 7.1 does not use the trivialization of the cotangent bundle. We can choose a diffeomorphism relating $M_\kappa(\hat{G})/U(1)$ and T^*G that does not break the left-right symmetry. The reason why we are not making such a symmetric choice (but we prefer the asymmetric one) is simple: It is because we want to arrive at the standard T^*G presentation of the WZW symplectic structure existing in the literature [20, 6].

It is instructive to evaluate the Poisson bracket of functions on T^*G with respect to the reduced form ω_{red} . It is convenient to use a short-hand notation $\langle \gamma_L, T^i \rangle \equiv \gamma_L^i$. Then the reduced form ω_{red} can be written as

$$\omega_{red} = +d\gamma_L^i \wedge R_{g^{-1}}^* t_i + \frac{1}{2}(\gamma_L^i f_i^{mn} + \kappa \rho(T^m, T^n)) R_{g^{-1}}^* t_m \wedge R_{g^{-1}}^* t_n, \quad (2.61)$$

where f_i^{mn} are the structure constants of the Lie algebra \mathcal{G} . This expression can be readily inverted to give the corresponding Poisson tensor Π_{red} :

$$\Pi_{red} = \frac{1}{2}(\kappa \rho(T^i, T^j) + \gamma_L^m f_m^{ij}) \frac{\partial}{\partial \gamma_L^i} \wedge \frac{\partial}{\partial \gamma_L^j} - \frac{\partial}{\partial \gamma_L^i} \wedge R_{g*} T^i. \quad (2.62)$$

Since we have that $\langle \nabla_G^L, \xi \rangle = R_{g*} \xi$, for $\xi \in \mathcal{G}$, we obtain from Π_{red} the following WZW Poisson brackets (cf. (7.41) -(7.43)):

$$\{\Phi_1(g), \Phi_2(g)\}_{red} = 0; \quad (2.63)$$

$$\{\Phi(g), \langle \gamma_L, \xi \rangle\}_{red} = \left(\frac{d}{ds} \right)_{s=0} \Phi(e^{s\xi} g) \equiv \langle \nabla_G^L \Phi, \xi \rangle; \quad (2.64)$$

$$\{\langle \gamma_L, \xi \rangle, \langle \gamma_L, \eta \rangle\}_{red} = \langle \gamma_L, [\xi, \eta] \rangle + \kappa \rho(\xi, \eta). \quad (2.65)$$

From this we obtain for the loop group case:

$$\begin{aligned} & \frac{1}{2\pi} \{ (T^\alpha, J_L(\sigma))_{\mathcal{G}_0}, (T^\beta, J_L(\sigma'))_{\mathcal{G}_0} \}_{red} = \\ & = ([T^\alpha, T^\beta], J_L(\sigma))_{\mathcal{G}_0} \delta(\sigma - \sigma') + \kappa (T^\alpha, T^\beta)_{\mathcal{G}_0} \partial_\sigma \delta(\sigma - \sigma'); \end{aligned} \quad (2.66)$$

$$\begin{aligned} & \frac{1}{2\pi} \{ (T^\alpha, J_R(\sigma))_{\mathcal{G}_0}, (T^\beta, J_R(\sigma'))_{\mathcal{G}_0} \}_{red} = \\ & = ([T^\alpha, T^\beta], J_R(\sigma))_{\mathcal{G}_0} \delta(\sigma - \sigma') - \kappa (T^\alpha, T^\beta)_{\mathcal{G}_0} \partial_\sigma \delta(\sigma - \sigma'); \end{aligned} \quad (2.67)$$

$$\frac{1}{2\pi} \{ g(\sigma), (T^\alpha, J_L(\sigma'))_{\mathcal{G}_0} \}_{red} = T^\alpha g(\sigma) \delta(\sigma - \sigma'); \quad (2.68)$$

$$\frac{1}{2\pi} \{g(\sigma), (T^\alpha, J_R(\sigma'))_{\mathcal{G}_0}\}_{red} = -g(\sigma) T^\alpha \delta(\sigma - \sigma'); \quad (2.69)$$

$$\frac{1}{2\pi} \{(T^\alpha, J_L(\sigma))_{\mathcal{G}_0}, (T^\beta, J_R(\sigma'))_{\mathcal{G}_0}\}_{red} = 0. \quad (2.70)$$

Here T^α is some element of the Lie algebra \mathcal{G}_0 , $g(\sigma)$ is understood as a matrix in some (typically fundamental) representation and $\delta(\sigma - \sigma')$ is the standard δ -function given by

$$\delta(\sigma - \sigma') = \frac{1}{2\pi} \sum_{n \in \mathbf{Z}} e^{in(\sigma - \sigma')}. \quad (2.71)$$

Upon to the (-2π) normalization (cf. the remark above concerning the normalization of the symplectic form), our reduced Poisson brackets (2.66)-(2.70) coincide with the Poisson brackets (2.4) of the reference [6] and thus they define the standard WZW symplectic structure.

Remarks:1) We should complete the list of the Poisson brackets (2.66)-(2.70) by the following "trivial" bracket:

$$\{g(\sigma) \otimes g(\sigma')\}_{red} = 0. \quad (2.72)$$

It will turn out that in the quasitriangular generalization of the WZW model such a bracket will not vanish.

2) It is important to note that the space derivate ∂_σ in the reduced Hamiltonian (2.59) was "born" in the process of the symplectic reduction. So we observe that field theoretic character of the WZW model is in a sense the fruit of the central extension.

Chapter 3

Chiral decomposition of the WZW model

It exists a sort of the square root of the dynamical structure of the standard WZW model. It is called the chiral WZW model [14, 13, 20] and it describes the dynamics of left (or right) movers independently. The full WZW model is then obtained by the appropriate glueing of the left and right chiral WZW theories. The goal of this chapter is to present the derivation of the chiral WZW model starting from the master model (1.1). We shall first decompose (1.1) into the chiral components (1.2) called the chiral master models and then we perform an appropriate two step symplectic reduction of the latters. We shall see that the result is indeed the standard chiral WZW model [14, 13, 20].

As it was often remarked [29, 4], the analogue of the chiral decomposition exists already at the level of finite-dimensional Lie groups. We shall devote a section to the description of this finite-dimensional story in order to set the technical, notational and ideological background for the more involved infinite-dimensional case.

3.1 Chiral geodesical model on G_0

3.1.1 Cartan decomposition

The geodesical model can be naturally associated with every Lie group possessing a biinvariant non-degenerate metric. In other words, it is required that an invariant symmetric non-degenerate \mathbf{R} -bilinear form $(\cdot, \cdot)_{\mathcal{G}}$ exists on the Lie algebra \mathcal{G} of the group G . In this section, we are going to study the case of a simple compact connected and simply connected group G_0 equipped with its standard Killing-Cartan form playing the role of $(\cdot, \cdot)_{\mathcal{G}_0}$.

In what follows, we shall introduce a map Υ_0 that identifies the dual \mathcal{G}_0^* of \mathcal{G}_0 with \mathcal{G}_0 itself via the bilinear form $(\cdot, \cdot)_{\mathcal{G}_0}$. Thus

$$\langle x^*, y \rangle = (\Upsilon_0(x^*), y)_{\mathcal{G}_0}, \quad x^* \in \mathcal{G}_0^*, \quad y \in \mathcal{G}_0. \quad (3.1)$$

Consider now a subspace $\Upsilon_0^{-1}(\mathcal{T})$ of \mathcal{G}_0^* , where \mathcal{T} is the Lie algebra of a chosen maximal torus \mathbf{T} in G_0 . We can view $\Upsilon_0^{-1}(\mathcal{T})$ also as the subspace of the cotangent space at the unit element of G_0 hence as the subgroup of T^*G_0 . In the latter case we shall denote $\Upsilon_0^{-1}(\mathcal{T})$ as \mathcal{A}^0 and we shall call it the Cartan subgroup of T^*G_0 . This terminology is not standard but it is very suitable for the purposes of this paper, in particular for the generalization to the loop group case.

Consider now subgroups $Norm(\mathcal{A}^0) \subset G_0$ and $Cent(\mathcal{A}^0) \subset G_0$ given by

$$Norm(\mathcal{A}^0) = \{w \in G_0, w\mathcal{A}^0w^{-1} \in \mathcal{A}^0\}; \quad (3.2)$$

$$Cent(\mathcal{A}^0) = \{w \in G_0, waw^{-1} = a \quad \text{if } a \in \mathcal{A}^0\}. \quad (3.3)$$

Here the group multiplication law is that of the cotangent bundle T^*G_0 . Clearly, $Cent(\mathcal{A}^0)$ is the normal subgroup of $Norm(\mathcal{A}^0)$.

Definition 3.1: Weyl group W is the factor group $Norm(\mathcal{A}^0)/Cent(\mathcal{A}^0)$.

Remark: The group $Cent(\mathcal{A}^0)$ is the maximal torus \mathbf{T} of G_0 .

The Weyl group acts \mathcal{T} or on \mathcal{A}^0 . The fundamental domains of this action on \mathcal{A}^0 are called (Weyl) chambers. One usually chooses one chamber which is then called the positive Weyl chamber and denoted as \mathcal{A}_+^0 .

It is the well-known fact that the G_0 -adjoint orbit of every element of \mathcal{G}_0 intersects the Cartan subalgebra \mathcal{T} of \mathcal{G}_0 (the diagonalization property in [8]). This fact, the trivializability of the cotangent bundle T^*G_0 and the definition of the Weyl chamber imply together the following theorem:

Theorem 3.2: (Cartan decomposition) Every element $K \in T^*G_0$ can be decomposed as

$$K = k_L \phi k_R^{-1}, \quad k_{L,R} \in G_0, \quad \phi \in \mathcal{A}_+^0. \quad (3.4)$$

The ambiguity of the decomposition is given by the simultaneous right multiplication of k_L and k_R by the same element of $\text{Cent}(\mathcal{A}^0) = \mathbf{T}$.

Proof: By the left trivialization, every element $K \in T^*G_0$ can be written as $K = g_L \beta_R$, where $g_L \in G_0$ and $\beta_R \in \mathcal{G}_0^*$. By diagonalization, β_R can be written as $\beta_R = k_R \phi k_R^{-1}$, for some $k_R \in G_0$ and $\phi \in \mathcal{A}^0$. By writing g_L as $k_L k_R^{-1}$ for certain $k_L \in G_0$ and by using the action of the Weyl group, we immediately arrive at the Cartan decomposition formula (3.4).

The theorem is proved. #

3.1.2 Standard chiral symplectic structure

Recall from section 7.2 that the symplectic potential θ on T^*G_0 can be simply expressed in the right trivialization $K = \beta_L g_R$ as

$$\theta = \langle \beta_L, \rho_{G_0} \rangle \equiv \langle \beta_L, dg_R g_R^{-1} \rangle. \quad (3.5)$$

The dynamical system characterized by the symplectic form $d\theta$ and by the Hamiltonian

$$H(K) = -(\beta_L(K), \beta_L(K))_{\mathcal{G}_0^*} \quad (3.6)$$

is called the standard geodesical model on G_0 . Recall that the form $(\cdot, \cdot)_{\mathcal{G}_0^*}$ is dual to $(\cdot, \cdot)_{\mathcal{G}_0}$. The latter form is defined by the restriction of the form the Killing-Cartan form $(\cdot, \cdot)_{\mathcal{G}_0^{\mathbb{C}}}$ to the compact real form \mathcal{G}_0 . As such, the form $(\cdot, \cdot)_{\mathcal{G}_0}$ is *negative* definite which explains the minus sign in the definition (3.6) of the Hamiltonian and also in the second order action:

$$S((g)) = -\frac{1}{4} \int d\tau (g^{-1} \frac{d}{d\tau} g, g^{-1} \frac{d}{d\tau} g)_{\mathcal{G}_0}. \quad (3.7)$$

Now consider a manifold $G_0 \times \mathcal{A}_+^0 \times G_0$; we shall denote its points as triples (k_L, ϕ, k_R) . The Cartan decomposition (3.4) then induces a natural map Ξ from this manifold into the cotangent double T^*G_0 . We can then pull-back the polarization form θ by the map Ξ . By noting that

$$\beta_L = k_L \phi k_L^{-1}, \quad g_R = k_L k_R^{-1}, \quad (3.8)$$

we obtain

$$\Xi^*\theta = \langle \text{Coad}_{k_L}\phi, dk_L k_L^{-1} + k_L dk_R^{-1} k_R k_L^{-1} \rangle = \langle \phi, k_L^{-1} dk_L \rangle - \langle \phi, k_R^{-1} dk_R \rangle. \quad (3.9)$$

Recall that ϕ is also the element of \mathcal{G}_0^* hence the pairing in (3.9) makes sense. We observe that the resulting form can be chirally decomposed in the left and right parts who talk to each other only via the variable ϕ . We can make the left and right form in (3.9) completely independent by means of the following construction

Consider a manifold $M_L = G_0 \times \mathcal{A}_+^0$. Its elements are couples (k_L, ϕ_L) and it is clearly a submanifold of T^*G_0 . We can pullback the symplectic potential θ on T^*G_0 to M_L by the map $(k_L, \phi_L) \rightarrow k_L \phi_L \in T^*G_0$, where the multiplication is in the sense of the group law in T^*G_0 . The result is clearly

$$\theta_L = \langle \phi_L, k_L^{-1} dk_L \rangle. \quad (3.10)$$

We shall prove soon that the form $d\theta_L$ on $G_0 \times \mathcal{A}_+^0$ is nondegenerate, hence it defines the symplectic structure.

Definition 3.3 : The manifold $M_L = G_0 \times \mathcal{A}_+^0$ equipped with the symplectic form $d\theta_L$ is referred to as the model space of the (simple compact etc.) group G_0 .

We have seen that we can obtain the symplectic structure on the model space by the simple pullback of the canonical symplectic form on T^*G_0 . We can show with the help of the Cartan decomposition that a sort of the "inverse" procedure is also possible. Indeed, consider a direct product $M_L \times M_R$ of two copies of the model space $M_L = G_0 \times \mathcal{A}_+^0$ and $M_R = G_0 \times \mathcal{A}_-^0$, where $\mathcal{A}_-^0 = -\mathcal{A}_+^0$. Equip the manifold $M_L \times M_R$ with a symplectic form

$$\omega_{L \times R} = d\theta_L + d\theta_R = d\langle \phi_L, k_L^{-1} dk_L \rangle + d\langle \phi_R, k_R^{-1} dk_R \rangle. \quad (3.11)$$

The cotangent bundle T^*G_0 with its canonical symplectic structure $\omega = d\theta$ can be obtained by an appropriate symplectic reduction of the symplectic manifold $(M_L \times M_R, \omega_{L \times R})$. Indeed, consider a submanifold of $M_L \times M_R$ obtained by equating $\phi_L + \phi_R = 0$. The form $\omega_{L \times R}$ restricted to this submanifold becomes just $\Xi^*\theta$. It is clearly degenerate, since by construction of the map Ξ , its kernel is given by vector fields generating the simultaneous right action of the maximal torus \mathbf{T} on k_L and k_R . By imposing the

equivalence $(k_L, k_R, \phi) \cong (k_L h, k_R h, \phi)$, where $h \in \mathbf{T}$, we obtain the reduced manifold. According to the Cartan decomposition theorem, the latter is nothing but T^*G_0 .

It turns out that the Hamiltonian (3.6) of the geodesical model on T^*G_0 can be also "descended" from a natural Hamiltonian on $M_L \times M_R$. The latter is given by the following formula

$$H_{L \times R} = H_L + H_R = -\frac{1}{2}(\phi_L, \phi_L)_{\mathcal{G}_0^*} - \frac{1}{2}(\phi_R, \phi_R)_{\mathcal{G}_0^*}. \quad (3.12)$$

Since $H_{L \times R}$ restricted to $\phi_L + \phi_R = 0$ is trivially invariant with respect to the maximal torus action $(k_L, k_R, \phi) \cong (k_L h, k_R h, \phi)$, it defines certain Hamiltonian on the reduced manifold T^*G_0 . In order to show that this is precisely the Hamiltonian of the geodesical flow in (3.6), it is sufficient to note, that

$$\begin{aligned} & (\beta_L(K), \beta_L(K))_{\mathcal{G}_0^*} = \\ & = (\beta_L(k_L \phi k_R^{-1}), \beta_L(k_L \phi k_R^{-1}))_{\mathcal{G}_0^*} = (k_L \phi k_L^{-1}, k_L \phi k_L^{-1})_{\mathcal{G}_0^*} = (\phi, \phi)_{\mathcal{G}_0^*}. \end{aligned} \quad (3.13)$$

3.1.3 Dynamical r-matrix

This section is devoted to the study of the chiral dynamical system defined on the model space M_L and characterized by the symplectic potential θ_L and the Hamiltonian H_L . This system has been proposed in [3] as the finite dimensional analogue of the chiral WZW model. We have seen in the previous paragraph that the geodesical model on G_0 admits the chiral decomposition in two chiral models. By this we mean that it can be defined by the symplectic reduction of the model on $M_L \times M_R$, characterized by the symplectic form $\omega_{L \times R}$ and by the Hamiltonian $H_{L \times R}$.

The chiral dynamics can be derived from the following action principle

$$S = \int d\tau [\langle \phi_L, k_L^{-1} \dot{k}_L \rangle + \frac{1}{2}(\phi_L, \phi_L)_{\mathcal{G}_0^*}], \quad (3.14)$$

where the dot indicates the time derivative. The equations of motion can be easily derived:

$$\frac{d}{d\tau}(Coad_{k_L} \phi_L) = 0, \quad P_{\mathcal{T}} k_L^{-1} \dot{k}_L = -\Upsilon_0(\phi_L), \quad (3.15)$$

where $P_{\mathcal{T}}$ denotes the orthogonal projection on \mathcal{T} and $\Upsilon_0 : \mathcal{G}_0^* \rightarrow \mathcal{G}_0$ is the map that identifies \mathcal{G}_0^* with \mathcal{G}_0 via the form $(\cdot, \cdot)_{\mathcal{G}_0}$.

From the first equation it follows

$$Coad_{k_L(\tau)}\phi_L(t) = Coad_{k_L(0)}\phi_L(0), \quad (3.16)$$

or

$$\phi_L(\tau) = Coad_{k_L^{-1}(\tau)k_L(0)}\phi_L(0). \quad (3.17)$$

This implies that $k_L(\tau)^{-1}k_L(0) \in \mathbf{T}$ and $\phi_L(\tau) = \phi_L(0)$, where \mathbf{T} is the maximal torus of G_0 . From this and the other equation (3.15), we finally obtain

$$k_L(\tau) = k_L(0) \exp[-\Upsilon_0(\phi_L(0))\tau]. \quad (3.18)$$

The only thing that changes in the treatment of the right model space M_R is the fact that $\phi_R(0) \in \mathcal{A}_-^0$. The solution of the right dynamical system is

$$k_R(\tau) = k_R(0) \exp[-\Upsilon_0(\phi_R(0))\tau]. \quad (3.19)$$

We glue the left and right system by identifying $\phi_L = -\phi_R$ which gives the standard geodesical motion on the group manifold

$$k(\tau) = k_L(\tau)k_R^{-1}(\tau) = k_L(0) \exp[-2\Upsilon_0(\phi_L(0))\tau]k_R^{-1}(0). \quad (3.20)$$

The symplectic form $d\theta_L$ can be easily inverted to give the Poisson bracket on the model space. Although this calculation was already detailed in the literature [4], we shall repeat it here as the simplest prototype of several similar but technically more involved computations that we shall be doing later on.

Recall that the Killing-Cartan form $(\cdot, \cdot)_{\mathcal{G}_0^{\mathbf{C}}}$ on $\mathcal{G}_0^{\mathbf{C}}$ is normalized in such a way that the square of the length of the longest root is equal to two. We pick an orthonormal basis $H^\mu \in i\mathcal{T}$ in the Cartan subalgebra $\mathcal{T}^{\mathbf{C}}$ of $\mathcal{G}_0^{\mathbf{C}}$ with respect to the Killing Cartan form $(\cdot, \cdot)_{\mathcal{G}_0^{\mathbf{C}}}$. Note that the elements H^μ are Hermitian hence they are not the elements of the Lie algebra \mathcal{T} of the maximal torus \mathbf{T} . Consider the root space decomposition of $\mathcal{G}_0^{\mathbf{C}}$:

$$\mathcal{G}_0^{\mathbf{C}} = \mathcal{T}^{\mathbf{C}} \oplus (\oplus_{\alpha \in \Phi} \mathbf{C}E^\alpha), \quad (3.21)$$

where α runs over the space Φ of all roots $\alpha \in \mathcal{T}^{\mathbf{C}*}$. The step generators E^α fulfil

$$[H^\mu, E^\alpha] = \alpha(H^\mu)E^\alpha, \quad (E^\alpha)^\dagger = E^{-\alpha}; \quad (3.22)$$

$$[E^\alpha, E^{-\alpha}] = \alpha^\vee, \quad [\alpha^\vee, E^{\pm\alpha}] = \pm 2E^{\pm\alpha}, \quad (E^\alpha, E^{-\alpha})_{\mathcal{G}_0^{\mathbb{C}}} = \frac{2}{|\alpha|^2}. \quad (3.23)$$

The element $\alpha^\vee \in i\mathcal{T}$ is called the coroot of the root α . Thus the basis of the *complex* vector space $\mathcal{G}_0^{\mathbb{C}}$ is (H^μ, E^α) , $\alpha \in \Phi$. The corresponding dual basis of $(\mathcal{G}_0^{\mathbb{C}})^*$ will be denoted as (h_μ, e_α) .

We want to invert the symplectic form on the model space M_L of the *compact real form* G_0 of the simple complex group $G_0^{\mathbb{C}}$. For this we need a basis on the Lie algebra \mathcal{G}_0 . We can construct such a basis in a canonical way from the basis (H^μ, E^α) on $\mathcal{G}_0^{\mathbb{C}}$. Set

$$T^\mu = iH^\mu, \quad B^\alpha = \frac{i}{\sqrt{2}}(E^\alpha + E^{-\alpha}), \quad C^\alpha = \frac{1}{\sqrt{2}}(E^\alpha - E^{-\alpha}). \quad (3.24)$$

The set $(T^\mu, B^\alpha, C^\alpha)$, is an orthogonal basis of the *real* vector space \mathcal{G}_0 with respect to $(\cdot, \cdot)_{\mathcal{G}_0}$. Note that α runs only over the positive roots in this context! The dual basis of \mathcal{G}_0^* will be denoted as $(t_\mu, b_\alpha, c_\alpha)$. Using the relation

$$k^{-1}dk = L_{k^{-1}}^* t_\mu \otimes T^\mu + \sum_{\alpha \in \Phi_+} (L_{k^{-1}}^* b_\alpha \otimes B^\alpha + L_{k^{-1}}^* c_\alpha \otimes C^\alpha), \quad (3.25)$$

we can write down the symplectic form $d\theta_L$ on M_L as

$$\begin{aligned} d\theta_L &= \langle d\phi \frown k^{-1}dk \rangle - \langle \phi, k^{-1}dk \wedge k^{-1}dk \rangle = \\ &= da^\mu \wedge L_{k^{-1}}^* t_\mu + \sum_{\alpha \in \Phi_+} \langle \phi, i\alpha^\vee \rangle L_{k^{-1}}^* b_\alpha \wedge L_{k^{-1}}^* c_\alpha. \end{aligned} \quad (3.26)$$

Here a^μ 's are defined by the expansion $\phi = a^\mu t_\mu$. In deriving (3.26), we have used the commutation relation

$$[B^\alpha, C^\alpha] = -i\alpha^\vee. \quad (3.27)$$

To make the formulas less cumbersome, we have also suppressed the index L on ϕ_L and k_L .

It is now very easy to invert the symplectic form $d\theta$. The corresponding Poisson tensor reads

$$\Pi_L = L_{k*} T^\mu \wedge \frac{\partial}{\partial a^\mu} - \sum_{\alpha \in \Phi_+} \frac{1}{\langle \phi, i\alpha^\vee \rangle} L_{k*} B^\alpha \wedge L_{k*} C^\alpha. \quad (3.28)$$

It is useful to give an explicit formula for the Poisson brackets of functions that can be obtained as matrix elements of representations of the group G_0 . Consider two finite-dimensional representations $\rho_i : G_0 \rightarrow \text{End}V_0$, $i = 1, 2$. The matrix element of the representation can be obtained as the function $\langle w^*, \rho_i(k)v \rangle$, where $v \in V_0$ and $w^* \in V_0^*$. The Poisson bracket of two such functions then reads

$$\begin{aligned} & \{\langle w_1^*, \rho_1(k)v_1 \rangle, \langle w_2^*, \rho_2(k)v_2 \rangle\}_{M_L} = \\ & = \langle (w_1^* \otimes w_2^*), (\rho_1(k) \otimes \rho_2(k))(\rho_1 \otimes \rho_2)(r_0(a))(v_1 \otimes v_2) \rangle, \end{aligned} \quad (3.29)$$

where

$$\begin{aligned} & r_0(a^\mu) = \\ & = \sum_{\alpha \in \Phi_+} \frac{-1}{\langle \phi, i\alpha^\vee \rangle} (B^\alpha \otimes C^\alpha - C^\alpha \otimes B^\alpha) = \sum_{\alpha \in \Phi_+} \frac{i|\alpha|^2}{2a^\mu \langle \alpha, H^\mu \rangle} E^\alpha \otimes E^{-\alpha}. \end{aligned} \quad (3.30)$$

The last equality in (3.30) follows from (3.24) and from the well-known relations

$$\alpha^\vee = \frac{2}{|\alpha|^2} \langle \alpha, H^\mu \rangle H^\mu, \quad \text{or} \quad i\alpha^\vee = \frac{2}{|\alpha|^2} \langle \alpha, H^\mu \rangle T^\mu. \quad (3.31)$$

Note that $r_0(a^\mu)$ is an a^μ -dependent element of $\mathcal{G}_0 \wedge \mathcal{G}_0$; it is called the dynamical r -matrix. It is to be contrasted with the standard r -matrix (cf. (1.9)) which does not depend on a^μ . Both standard and dynamical r -matrices have to satisfy some consistency conditions if the Poisson brackets based on them are to satisfy the Jacobi identities. Those conditions are called, respectively, the Yang-Baxter and the dynamical Yang-Baxter equations. We do not worry about the Jacobi identity here because we know a priori that the symplectic form $d\theta_L$ is closed.

Physicists use the so called matrix Poisson brackets (cf. e.g. [5, 19]) in order to make the expressions like (3.29) more transparent. For simplicity, let us consider the case where $\rho_1 = \rho_2$ and choose some basis of the representation space V_0 . Then the Poisson bracket of two matrix valued functions A_{ij} and B_{kl} is written as

$$\{A_{ij}, B_{kl}\} \equiv \{A \otimes B\}_{ik,jl}. \quad (3.32)$$

With such a notation, we can write the brackets (3.29) of the matrix elements in the following matrix form:

$$\{\rho(k) \otimes \rho(k)\}_{M_L} = (\rho(k) \otimes \rho(k))\rho(r_0(a^\mu)). \quad (3.33)$$

Even more often, people use a notation where the dependence on the representation ρ is explicitly suppressed but tacitly assumed, i.e.

$$\{k \otimes k\}_{M_L} = (k \otimes k)r_0(a^\mu). \quad (3.34)$$

The Poisson bracket between the k and a variable can also be written in the matrix form as follows

$$\{k, a^\mu\}_{M_L} = kT^\mu. \quad (3.35)$$

We complete this section with the commutation relation of the moment maps generating the left action of G_0 on $M_L = G_0 \times \mathcal{A}_+^0$. The symplectic potential θ_L (hence the symplectic form $d\theta_L$) is clearly invariant with respect to the left multiplication by any $k_0 \in G_0$. Consider the infinitesimal vector field $V = R_{k_L*}T$ on M_L corresponding to the left action of a generator $T \in \mathcal{G}_0$. As usual (cf. also (4.87)), the corresponding moment map $\langle M, T \rangle$ is defined by the relation

$$-i_V d\theta_L \equiv d\theta_L(., V) = d\langle M, T \rangle. \quad (3.36)$$

The invariance of the symplectic potential θ_L means the vanishing of its Lie derivative with respect to V . In other words:

$$(i_V d + di_V)\theta_L = 0. \quad (3.37)$$

From this relation, it immediately follows that

$$\langle M, T \rangle = i_V \theta_L = \langle \phi_L, L_{k_L^{-1}*} R_{k_L*} T \rangle = \langle \text{Coad}_{k_L} \phi_L, T \rangle = \langle \beta_L(k_L \phi_L), T \rangle. \quad (3.38)$$

Recall that $\beta_L(K)$ is the \mathcal{G}_0^* -valued map defined by the decomposition $K = \beta_L(K)g_R(K)$. From (3.36), one can immediately infer:

$$\Pi_L(., d\langle M, T \rangle) = R_{k*}T. \quad (3.39)$$

Eqs. (3.38) and (3.39) imply that for any function $f(k_L, \phi_L)$ on M_L it holds

$$\{f(k_L, \phi_L), \langle \beta_L(k_L \phi_L), T \rangle\}_{M_L} = \langle \nabla_{G_0}^L f, T \rangle, \quad (3.40)$$

where the differential operator $\nabla_{G_0}^L$ is defined in (7.54). In particular, by remarking that $\beta_L(k_L \phi_L) = k_L \phi_L k_L^{-1}$, we obtain

$$\{\langle \beta_L(k_L \phi_L), x \rangle, \langle \beta_L(k_L \phi_L), y \rangle\}_{M_L} = \langle \beta_L(k_L \phi_L), [x, y] \rangle, \quad x, y \in \mathcal{G}_0. \quad (3.41)$$

Of course, the same relation can be directly obtained from the fact that $\beta_L(k_L \phi_L)$ is the moment map generating the left action of \mathcal{G}_0 on M_L .

3.2 Chiral decomposition of the master model

3.2.1 Affine Cartan decomposition

Usually people derive the standard WZW left-right decomposition by using the equations of motion of the full WZW theory [20]. The solutions of these equations of motion split into left and right movers. Because the phase space of a dynamical system can be identified with its space of solutions, one infers that the phase space itself can be split in its chiral parts. The corresponding symplectic forms on the chiral parts have been derived by Gawędzki [29].

Here we shall show how the standard chiral WZW dynamics [29] emerges from the perspective of the master model (1.1). We do not start with the equations of motion. Instead, we shall consider a Cartan-like decomposition of the cotangent bundle of the centrally biextended loop group \tilde{G} . This will give the left-right splitting without the use of the field equations.

Denote \mathcal{T} the Lie algebra of the maximal torus \mathbf{T} of the simple compact group G_0 . Clearly, \mathcal{T} can be interpreted also as the subalgebra of the loop group algebra $L\mathcal{G}_0$ consisting of the constant maps from S^1 into \mathcal{T} . In what follows, we shall use the same notation for \mathcal{T} being the subalgebra of \mathcal{G}_0 or of $L\mathcal{G}_0$. Now we consider following subalgebra of $\tilde{\mathcal{G}}$:

$$\tilde{\mathcal{T}} = \mathbf{R}\tilde{T}^0 + i(\mathcal{T}) + \mathbf{R}\tilde{T}^\infty. \quad (3.42)$$

Their elements are triples (iX, ξ_0, ix) , $X, x \in \mathbf{R}$, $\xi_0 \in \mathcal{G}_0 \subset L\mathcal{G}_0$ in the terminology of Section 2.1. The subalgebra $\tilde{\mathcal{T}}$ can be mapped by $\tilde{\Upsilon}^{-1}$ to the subgroup $\tilde{\Upsilon}^{-1}(\tilde{\mathcal{T}})$ of $T^*\tilde{G}$, since $\tilde{\mathcal{G}}^*$ can be identified with the cotangent space at the unit element of the group \tilde{G} . Here, as usual, the identification map $\tilde{\Upsilon} : \tilde{\mathcal{G}}^* \rightarrow \tilde{\mathcal{G}}$ is induced by the invariant bilinear form (2.10) on $\tilde{\mathcal{G}}$.

Definition 3.4: $\tilde{\Upsilon}^{-1}(\tilde{\mathcal{T}}) \equiv \tilde{\mathcal{A}}$ is called the Cartan subgroup of the cotangent space $T_{\tilde{e}}^*\tilde{G}$ at the unit element \tilde{e} of \tilde{G} . In the terminology of Section 2.1, their elements are triples $\tilde{\phi} = (a^0, \phi, a^\infty)^*$, where $\phi \in \tilde{\Upsilon}^{-1}(\mathcal{T})$, $a^0, a^\infty \in \mathbf{R}$. Of course, $\tilde{\mathcal{A}}$ can be also interpreted as the subalgebra of the Lie algebra $\tilde{\mathcal{D}}$ of the group $\tilde{D} = T^*\tilde{G}$. We shall also define two subalgebras of $\tilde{\mathcal{A}}$ denoted $\hat{\mathcal{A}}$ and \mathcal{A} , the former is spanned by elements having $a^0 = 0$ and the latter by those having $a^0 = a^\infty = 0$.

Consider now subgroups $Norm(\tilde{\mathcal{A}}) \subset \hat{G}$ and $Cent(\tilde{\mathcal{A}}) \subset \hat{G}$ given by

$$Norm(\tilde{\mathcal{A}}) = \{\hat{w} \in \hat{G}, \widetilde{Coad}_{\hat{w}}\tilde{\mathcal{A}} \in \tilde{\mathcal{A}}\}; \quad (3.43)$$

$$Cent(\tilde{\mathcal{A}}) = \{\hat{w} \in \hat{G}, \widetilde{Coad}_{\hat{w}}\tilde{\phi} = \tilde{\phi} \text{ if } \tilde{\phi} \in \tilde{\mathcal{A}}\}. \quad (3.44)$$

Clearly, $Cent(\tilde{\mathcal{A}})$ is a normal subgroup of $Norm(\tilde{\mathcal{A}})$.

Definition 3.5: Affine Weyl group \tilde{W} is the factor group $Norm(\tilde{\mathcal{A}})/Cent(\tilde{\mathcal{A}})$.

Remark: We shall see soon that the group $Cent(\tilde{\mathcal{A}})$ is nothing but the direct product $\mathbf{T} \times U(1)$, where \mathbf{T} is the maximal torus of G_0 and $U(1)$ is the central circle subgroup of $\hat{G} = \widehat{LG}_0$. It is important to realize in this context that the circle bundle over $\mathbf{T} \subset LG_0$ is trivial hence \mathbf{T} can be embedded in \hat{G} .

The coadjoint action of \hat{G} on $\tilde{\mathcal{A}}$ is given by the formula (2.23):

$$\begin{aligned} \widetilde{Coad}_{\hat{g}}\tilde{\phi} &= \widetilde{Coad}_{\hat{g}}(a^0, \phi, a^\infty)^* = \\ &= (a^0 + \langle \phi, g^{-1}\partial g \rangle + \frac{1}{2}a^\infty(g^{-1}\partial g, g^{-1}\partial g)_{\mathcal{G}}, Coad_g\phi + a^\infty\Upsilon^{-1}(\partial g g^{-1}), a^\infty)^*. \end{aligned} \quad (3.45)$$

Consider now an element $h(\sigma) = e^{iv\sigma}$ from the coroot group $Hom(U(1), \mathbf{T})$. We have

$$\begin{aligned} \widetilde{Coad}_{\hat{h}}(a^0, \phi, a^\infty)^* &= \\ &= (a^0 + \langle \phi, iv \rangle + \frac{1}{2}a^\infty(iv, iv)_{\mathcal{G}}, \phi + a^\infty\Upsilon^{-1}(iv), a^\infty)^*. \end{aligned} \quad (3.46)$$

Here \hat{h} is some π^{-1} -lift of $h(\sigma)$ into \hat{G} and $v \in \mathcal{H}(= i\mathcal{T})$ is an σ -independent element called the coroot which corresponds to the element $h(\sigma)$ in $Hom(U(1), \mathbf{T})$. This correspondence is clearly one-to-one and for this reason the coroot group $Hom(U(1), \mathbf{T})$ is often viewed as the coroot lattice in $i\mathcal{T}$ or in \mathcal{T} .

The inspection of the formulae (3.45) and (3.46) tells us what is the affine Weyl group. It is the semidirect product of the standard Weyl group of G_0 (which can be also naturally embedded in \hat{G}) and of the coroot group $Hom(U(1), \mathbf{T})$. By construction, the affine Weyl group acts on $\tilde{\mathcal{A}}$. Elements of \tilde{W} can be represented by the elements of \hat{G} . For the elements of the

ordinary Weyl group W , standard representation by the elements G_0 can be chosen. For the elements of the coroot lattice, the representation is evident since every element of $Hom(U(1), \mathbf{T})$ can be viewed by definition as the element of the loop group $L\mathbf{T}$. By the way, the fact that $Cent(\hat{A}) = \mathbf{T} \times U(1)$ follows directly from the formulae (3.45) and (3.46).

We immediately realize from (3.46), that the affine Weyl group acts not only on $\tilde{\mathcal{A}}$, but also on $\hat{\mathcal{A}}$ and \mathcal{A} . However, in the latter case the action depends on A^∞ as on the parameter. The fundamental domains of this action on \mathcal{A} are called alcoves. Consider the decomposition of the positive Weyl chamber into alcoves. The alcove attached to the origin (zero) of this positive Weyl chamber is referred to as the fundamental alcove $\mathcal{A}_+^{a^\infty}$. It clearly depends on a^∞ . Now an element from $\tilde{\mathcal{A}}$ of the form $(a^0, \phi, a^\infty)^*$ is said to be in $\tilde{\mathcal{A}}_+$, if ϕ is in the fundamental alcove $\mathcal{A}_+^{a^\infty}$. Hoping not to create too much confusion, we shall call $\tilde{\mathcal{A}}_+$ the fundamental alcove, too.

After this preliminary discussion we can now state the important theorem.

Theorem 3.6: Every element $\tilde{K} \in T^*\tilde{G}$ can be decomposed as

$$\tilde{K} = \tilde{k}_L \tilde{\phi} \tilde{k}_R^{-1}, \quad \tilde{k}_{L,R} \in \tilde{G}, \quad \tilde{\phi} \in \tilde{\mathcal{A}}_+. \quad (3.47)$$

The ambiguity of the decomposition is given by the simultaneous right multiplication of \tilde{k}_L and \tilde{k}_R by the same element of $Cent(\hat{A}) \times \mathbf{R}_S = \exp \tilde{T}$.

Proof: Obviously, we must prove that every element of $\tilde{\mathcal{G}}^*$ can be connected to some element in $\tilde{\mathcal{A}}_+$ by the coadjoint action of \tilde{G} . In other words, for every $(C, \gamma, c)^* \in \tilde{\mathcal{G}}^*$ it exists $\tilde{k}_L \in \tilde{G}$ and $(a^0, \phi, a^\infty)^* \in \tilde{\mathcal{A}}_+$ such that

$$\widetilde{Coad}_{\tilde{k}_L}(a^0, \phi, a^\infty)^* = (C, \gamma, c)^*. \quad (3.48)$$

In fact, it turns out that already $\hat{k}_L \in \hat{G}$ does the job. Indeed, we have

$$\begin{aligned} &= \widetilde{Coad}_{\hat{k}_L}(a^0, \phi, a^\infty)^* = \\ &(a^0 + \langle \phi, k_L^{-1} \partial k_L \rangle + \frac{1}{2} a^\infty (k_L^{-1} \partial k_L, k_L^{-1} \partial k_L)_{\mathcal{G}}, Coad_{k_L} \phi + a^\infty \Upsilon^{-1}(\partial k_L k_L^{-1}), a^\infty)^*, \end{aligned} \quad (3.49)$$

where we remind our convention $k_L = \pi(\hat{k}_L)$ if both \hat{k}_L and k_L are present in the same formula. Thus we have to show that every γ can be written as

$$\Upsilon(\gamma) = k_L \Upsilon(\phi) k_L^{-1} + a^\infty \partial_\sigma k_L k_L^{-1}, \quad (3.50)$$

where ϕ is in the fundamental alcove $\mathcal{A}_+^{a^\infty}$. Define

$$V(\sigma) = \overleftarrow{P} \exp \int_0^\sigma \frac{\Upsilon(\gamma)(\sigma)}{a^\infty} d\sigma. \quad (3.51)$$

The monodromy $V(2\pi)$ is the element of the compact group G_0 , hence it can be diagonalized [9, 20] as

$$V(2\pi) = k_0 e^{2\pi\rho} k_0^{-1}, \quad (3.52)$$

where k_0 is in G_0 and ρ in the alcove $\Upsilon(\mathcal{A}_+^1)$. Define now k_L and ϕ as follows

$$k_L(\sigma) = V(\sigma) k_0 \exp(-\rho\sigma), \quad \phi = a^\infty \Upsilon^{-1}(\rho). \quad (3.53)$$

It can be easily check that the pair (k_L, ϕ) defined in this way satisfies (3.50). Thus we see that the elements \tilde{k}_L , \tilde{k}_R and $\tilde{\phi}$ from the statement of the theorem always exist, moreover, $\tilde{\phi}$ is unique. It then easily follows that ambiguity of the choice of \tilde{k}_L and \tilde{k}_R is given by the simultaneous right multiplication by an element from $\exp \tilde{T}$.

The theorem is proved.

#

3.2.2 Affine model space

We wish to construct the phase space of the loop group chiral WZW model. The first part of the exposition of this section will follow the spirit of the Section 3.1. Indeed, we shall equip the affine model space $\tilde{M}_L = \tilde{G} \times \tilde{A}_+$ with a symplectic structure by taking the pull back of the canonical symplectic form on the cotangent bundle $T^*\tilde{G}$ of the *centrally biextended* loop group. We shall also write down the natural Hamiltonian thus constructing the chiral geodesical model on the affine Kac-Moody group \tilde{G} . Then we shall make steps which are not rooted in Section 3.1; namely, we perform the symplectic reduction of that chiral master model (down to the chiral WZW model).

Recall from section 7.2 that the symplectic potential $\tilde{\theta}$ on $T^*\tilde{G}$ can be simply expressed in the right trivialization $\tilde{K} = \tilde{\beta}_L \tilde{g}$ as

$$\tilde{\theta} = \langle \tilde{\beta}_L, d\tilde{g}\tilde{g}^{-1} \rangle. \quad (3.54)$$

The dynamical system characterized by the symplectic form $d\tilde{\theta}$ and by the Hamiltonian

$$\tilde{H}(\tilde{K}) = -\frac{1}{\kappa}(\tilde{\beta}_L(\tilde{K}), \tilde{\beta}_L(\tilde{K}))_{\tilde{\mathcal{G}}^*} \quad (3.55)$$

is nothing but the master model (1.1).

Consider the affine model space $\tilde{M}_L = \tilde{G} \times \tilde{\mathcal{A}}_+$. Its elements are couples $(\tilde{k}_L, \tilde{\phi}_L)$ and it is clearly the submanifold of $T^*\tilde{G}$. We can pullback the symplectic potential $\tilde{\theta}$ on $T^*\tilde{G}$ to \tilde{M}_L by the map $(\tilde{k}_L, \tilde{\phi}_L) \rightarrow \tilde{k}_L \tilde{\phi}_L \in T^*\tilde{G}$, where the group multiplication law is considered in the sense of $T^*\tilde{G}$. The result is clearly

$$\tilde{\theta}_L = \langle \tilde{\phi}_L, \tilde{k}_L^{-1} d\tilde{k}_L \rangle. \quad (3.56)$$

The form $d\tilde{\theta}_L$ on $\tilde{G} \times \tilde{\mathcal{A}}_+$ will turn out to be non-degenerate, hence it defines a symplectic structure.

We have seen that we can obtain the symplectic structure on the affine model space by the simple pullback of the canonical symplectic form on $T^*\tilde{G}$. We can show with the help of the affine Cartan decomposition that a sort of the "inverse" procedure is also possible. Indeed, consider the direct product $\tilde{M}_L \times \tilde{M}_R$ of two copies of the model space $\tilde{M}_L = \tilde{G} \times \tilde{\mathcal{A}}_+$ and $\tilde{M}_R = \tilde{G} \times \tilde{\mathcal{A}}_-$, where $\tilde{\mathcal{A}}_- = -\tilde{\mathcal{A}}_+$. Equip the manifold $\tilde{M}_L \times \tilde{M}_R$ with a symplectic form

$$\tilde{\omega}_{L \times R} = d\tilde{\theta}_L + d\tilde{\theta}_R = d\langle \tilde{\phi}_L, \tilde{k}_L^{-1} d\tilde{k}_L \rangle + d\langle \tilde{\phi}_R, \tilde{k}_R^{-1} d\tilde{k}_R \rangle. \quad (3.57)$$

The cotangent bundle $T^*\tilde{G}$ with its canonical symplectic structure $\tilde{\omega} = d\tilde{\theta}$ can be obtained by the appropriate symplectic reduction of the symplectic manifold $(\tilde{M}_L \times \tilde{M}_R, \tilde{\omega}_{L \times R})$ induced by equating $\tilde{\phi}_L + \tilde{\phi}_R = 0$. The argument proving this statement is step by step identical to the finite dimensional argument of Section 3.2 and we shall not repeat it here.

We have just shown that the symplectic structure of the geodesical model on \tilde{G} can be obtained by the symplectic reduction of the product of two affine model spaces. It turns out that also the Hamiltonian on $T^*\tilde{G}$ can be descended from a Hamiltonian on $\tilde{M}_L \times \tilde{M}_R$. The latter is given by

$$\tilde{H}_{L \times R} = \tilde{H}_L + \tilde{H}_R = -\frac{1}{2\kappa}(\tilde{\phi}_L, \tilde{\phi}_L)_{\tilde{\mathcal{G}}^*} - \frac{1}{2\kappa}(\tilde{\phi}_R, \tilde{\phi}_R)_{\tilde{\mathcal{G}}^*}. \quad (3.58)$$

Here $(\cdot, \cdot)_{\tilde{\mathcal{G}}^*}$ is the form (2.22).

The symplectic reduction from $\tilde{M}_L \times \tilde{M}_R$ to $T^*\tilde{G}$ is governed by the moment maps $\tilde{\phi}_L + \tilde{\phi}_R$. It generates the simultaneous right action of the

group $\exp \tilde{\mathcal{T}} \equiv \mathbf{T} \times U(1) \times \mathbf{R}_S$ on \tilde{k}_L and \tilde{k}_R . The Hamiltonian $\tilde{H}_{L \times R}$ is invariant with respect to this action hence it defines the Hamiltonian \tilde{H} on $T^*\tilde{G}$ given by

$$\begin{aligned} \tilde{H} &= -\frac{1}{2\kappa}(\widetilde{Coad}_{\tilde{k}_L} \tilde{\phi}_L, \widetilde{Coad}_{\tilde{k}_L} \tilde{\phi}_L)_{\tilde{G}^*} - \frac{1}{2\kappa}(\widetilde{Coad}_{\tilde{k}_L} \tilde{\phi}_L, \widetilde{Coad}_{\tilde{k}_L} \tilde{\phi}_L)_{\tilde{G}^*} = \\ &= -\frac{1}{\kappa}(\tilde{\beta}_L(\tilde{k}_L \tilde{\phi} \tilde{k}_R^{-1}), \tilde{\beta}_L(\tilde{k}_L \tilde{\phi} \tilde{k}_R^{-1}))_{\tilde{G}^*} = -\frac{1}{\kappa}(\tilde{\beta}_L(\tilde{K}), \tilde{\beta}_L(\tilde{K}))_{\tilde{G}^*}. \end{aligned} \quad (3.59)$$

Thus we have recovered the Hamiltonian \tilde{H} given by the formula (3.55).

In what follows, we are therefore going to study the chiral model on \tilde{M}_L given by the action

$$\tilde{S}_L(\tilde{k}_L, \tilde{\phi}_L) = \int d\tau [\langle \tilde{\phi}_L, \tilde{k}_L^{-1} \frac{d}{d\tau} \tilde{k}_L \rangle + \frac{1}{2\kappa}(\tilde{\phi}_L, \tilde{\phi}_L)_{\tilde{G}^*}]. \quad (3.60)$$

So far we have learned that the master model (1.1) on \tilde{G} admits the chiral decomposition into two chiral models (3.60). By this we mean that it can be defined by the symplectic reduction of the model defined on $\tilde{M}_L \times \tilde{M}_R$ and characterized by the symplectic form $\tilde{\omega}_{L \times R}$ and by the Hamiltonian $\tilde{H}_{L \times R}$. Combining this fact with the results of Section 2.2, we learn that the standard WZW model can be produced by glueing the two models (3.60) and then performing the symplectic reduction. Next we shall show that we arrive at the same result (WZW model) if we first perform a simple symplectic reduction at the chiral level (3.60) and then glue two such reduced (chiral WZW) models.

3.2.3 Chiral reduction to the first floor

In order to perform the first step of the reduction, we should evaluate the standard (Abelian) moment maps generating the left action of \tilde{G} on \tilde{M}_L . The symplectic potential $\tilde{\theta}_L$ (hence the symplectic form $d\tilde{\theta}_L$) is clearly invariant with respect to the left multiplication by any $\tilde{k}_0 \in \tilde{G}$. Consider an infinitesimal vector field $\tilde{V} = R_{\tilde{k}_L^*} \tilde{T}$ on \tilde{M}_L corresponding to the left action of a generator $\tilde{T} \in \tilde{G}$. As usual, the corresponding moment map $\langle \tilde{M}, \tilde{T} \rangle$ is defined by the relation

$$-i_{\tilde{V}} d\tilde{\theta}_L \equiv d\tilde{\theta}_L(., \tilde{V}) = d\langle \tilde{M}, \tilde{T} \rangle. \quad (3.61)$$

The invariance of the symplectic potential $\tilde{\theta}_L$ means the vanishing of its Lie derivative with respect to \tilde{V} . In other words:

$$(i_{\tilde{V}}d + d i_{\tilde{V}})\tilde{\theta}_L = 0. \quad (3.62)$$

From this relation, it immediately follows that

$$\langle \tilde{M}, \tilde{T} \rangle = i_{\tilde{V}}\tilde{\theta}_L = \langle \tilde{\phi}_L, L_{\tilde{k}_L^{-1}*} R_{\tilde{k}_L*} \tilde{T} \rangle = \langle \widetilde{Coad_{\tilde{k}_L}} \tilde{\phi}_L, \tilde{T} \rangle = \langle \tilde{\beta}_L(\tilde{k}_L \tilde{\phi}_L), \tilde{T} \rangle. \quad (3.63)$$

Since $\tilde{\beta}_L(\tilde{k}_L \tilde{\phi}_L) = \widetilde{Coad_{\tilde{k}_L}} \tilde{\phi}_L$ is the moment map of the standard Hamiltonian left action of \tilde{G} on \tilde{M}_L , we infer immediately the following Poisson brackets of its coefficient functions on \tilde{M}_L :

$$\{\langle \tilde{\beta}_L(\tilde{k}_L \tilde{\phi}_L), \tilde{x} \rangle, \langle \tilde{\beta}_L(\tilde{k}_L \tilde{\phi}_L), \tilde{y} \rangle\}_{\tilde{M}_L} = \langle \tilde{\beta}_L(\tilde{k}_L \tilde{\phi}_L), [\tilde{x}, \tilde{y}] \rangle, \quad \tilde{x}, \tilde{y} \in \tilde{\mathcal{G}}. \quad (3.64)$$

The particular case of these Poisson brackets will play the important role in what follows:

$$\begin{aligned} & \{\langle \tilde{\beta}_L(\tilde{k}_L \tilde{\phi}_L), (0, \xi, 0) \rangle, \langle \tilde{\beta}_L(\tilde{k}_L \tilde{\phi}_L), (0, \eta, 0) \rangle\}_{\tilde{M}_L} \\ &= \langle \tilde{\beta}_L(\tilde{k}_L \tilde{\phi}_L), (0, [\xi, \eta], 0) \rangle + a_L^\infty \rho(\xi, \eta), \quad \xi, \eta \in \mathcal{G}. \end{aligned} \quad (3.65)$$

Here ρ is the loop group cocycle (7.16) and we have used the fact that

$$\langle \tilde{\beta}_L(\tilde{k}_L \tilde{a}_L), \tilde{T}^\infty \rangle = a_L^\infty. \quad (3.66)$$

a_L^∞ is defined by the decomposition (for the notation cf. Section 3.2.1)

$$\tilde{\phi}_L = (a_L^0, \phi_L, a_L^\infty)^*. \quad (3.67)$$

Note also that $\tilde{k}_L \tilde{\phi}_L$ is the product in the sense of $T^*\tilde{G}$.

Now we parametrize $\tilde{k}_L = u \hat{k}_L$, $u = \exp s \tilde{T}^0$ and rewrite the action (3.60) as

$$\tilde{S}_L(s, \hat{k}_L, \tilde{\phi}_L) = \int d\tau [\langle \widetilde{Coad_{\hat{k}_L}} \tilde{\phi}_L, \tilde{T}^0 \rangle \frac{ds}{d\tau} + \langle \hat{\phi}_L, \hat{k}_L^{-1} \frac{d}{d\tau} \hat{k}_L \rangle + \frac{1}{2\kappa} (\tilde{\phi}_L, \tilde{\phi}_L)_{\tilde{\mathcal{G}}^*}]. \quad (3.68)$$

Of course, $\langle \widetilde{Coad_{\hat{k}_L}} \tilde{\phi}_L, \tilde{T}^0 \rangle \equiv \tilde{\beta}_L^0$ is the moment map generating the infinitesimal left action of \tilde{T}^0 and we put

$$\hat{\phi}_L = (0, \phi_L, a_L^\infty)^*. \quad (3.69)$$

Now we introduce the set of coordinates $(s, \tilde{\beta}_L^0, \hat{k}_L, \hat{\phi}_L)$ and, using the formula (2.23), we calculate

$$\tilde{\beta}_L^0 = \langle \widetilde{Coa}_{\hat{k}_L} \tilde{\phi}_L, \tilde{T}^0 \rangle = a_L^0 + \langle \phi_L, k_L^{-1} \partial k_L \rangle + \frac{1}{2} a_L^\infty (k_L^{-1} \partial k_L, k_L^{-1} \partial k_L)_{\mathcal{G}}. \quad (3.70)$$

As usual, here $k_L = \pi(\hat{k}_L)$. The action (3.6.8) finally becomes

$$\begin{aligned} \tilde{S}_L(s, \tilde{\beta}_L^0, \hat{k}_L, \hat{a}_L) = & \int d\tau [\tilde{\beta}_L^0 \frac{ds}{d\tau} - \frac{1}{\kappa} \tilde{\beta}_L^0 a_L^\infty + \\ & + \langle \hat{\phi}_L, \hat{k}_L^{-1} \frac{d}{d\tau} \hat{k}_L \rangle + \frac{1}{2\kappa} (\phi_L, \phi_L)_{\mathcal{G}^*} + \frac{a_L^\infty}{\kappa} \langle \phi_L, k_L^{-1} \partial k_L \rangle + \frac{(a_L^\infty)^2}{2\kappa} (k_L^{-1} \partial k_L, k_L^{-1} \partial k_L)_{\mathcal{G}}.] \end{aligned} \quad (3.71)$$

The Hamiltonian $\tilde{H}_L = -\frac{1}{2\kappa} (\tilde{\phi}_L, \tilde{\phi}_L)_{\tilde{\mathcal{G}}^*}$ is obviously invariant with respect to the action of \tilde{T}^0 on \tilde{M}_L , hence it Poisson-commutes with the moment map $\tilde{\beta}_L^0$ and we can consistently set $\tilde{\beta}_L^0$ to zero in (3.71). This constitutes the first step of the symplectic reduction.

3.2.4 Ground floor: standard chiral WZW model

Recall that the first step of the symplectic reduction from the chiral master model (3.60) gave the result

$$\begin{aligned} \hat{S}_L(\hat{k}_L, \hat{\phi}_L) = & \int d\tau [\langle \hat{\phi}_L, \hat{k}_L^{-1} \frac{d}{d\tau} \hat{k}_L \rangle + \\ & + \frac{1}{2\kappa} (\phi_L, \phi_L)_{\mathcal{G}^*} + \frac{a_L^\infty}{\kappa} \langle \phi_L, k_L^{-1} \partial k_L \rangle + \frac{(a_L^\infty)^2}{2\kappa} (k_L^{-1} \partial k_L, k_L^{-1} \partial k_L)_{\mathcal{G}}.] \end{aligned} \quad (3.72)$$

This first floor chiral theory is formulated on the phase space $\hat{M}_L = \hat{G} \times \hat{\mathcal{A}}_+$ that we shall call the reduced affine model space. Recall that the elements of the alcove $\hat{\mathcal{A}}_+$ have the form $(0, \phi_L, a^\infty)$, where $\phi_L \in \mathcal{A}_+^{a^\infty}$. The Hamiltonian \hat{H}_L is given by the collection of terms in (3.72) depending on κ . On the other hand, the symplectic potential is independent on κ .

In order to perform the second step of the symplectic reduction, it will be convenient to express the symplectic form

$$\hat{\omega}_L = d\hat{\theta}_L = d\langle \hat{\phi}_L, \hat{k}_L^{-1} d\hat{k}_L \rangle \quad (3.73)$$

in some basis of the Lie algebra $\hat{\mathcal{G}} = \widehat{L\mathcal{G}_0}$. The convenient basis can be obtained by injecting a basis of $\mathcal{G} = L\mathcal{G}_0$ into $\hat{\mathcal{G}}$ by the map ι and adding the generator \hat{T}^∞ . The basis of $L\mathcal{G}_0$, in turn, can be naturally constructed from the canonical Cartan-Weyl basis of the complexified Lie algebra $L\mathcal{G}_0^{\mathbb{C}}$. The step generators of the latter are of the form

$$E^\alpha e^{in\sigma} \equiv E_n^\alpha, \quad n \in \mathbf{Z}, \quad H^\mu e^{in\sigma} \equiv H_n^\mu, \quad n \in \mathbf{Z}, n \neq 0, \quad (3.74)$$

where E^α, H^μ is the basis of $\mathcal{G}_0^{\mathbb{C}}$ (cf. Section 3.1.3) and σ is the loop parameter. In what follows, we shall often denote a generic element of the set (3.74) as $E^{\hat{\alpha}}$, where $\hat{\alpha} \in \hat{\Phi}$ stands for the corresponding labels (α, n) or $(\mu, n \neq 0)$. If $\hat{\alpha}$ is such that α, μ are arbitrary and $n > 0$, or $\alpha \in \Phi_+$ and $n = 0$, we say that $\hat{\alpha} \in \hat{\Phi}_+$. The basis of the Lie algebra $L\mathcal{G}_0$ can be then chosen as $(T^\mu, B^{\hat{\alpha}}, C^{\hat{\alpha}})$, $\hat{\alpha} \in \hat{\Phi}_+$ where

$$T^\mu = iH^\mu, \quad B^{\hat{\alpha}} = \frac{i}{\sqrt{2}}(E^{\hat{\alpha}} + E^{-\hat{\alpha}}), \quad C^{\hat{\alpha}} = \frac{1}{\sqrt{2}}(E^{\hat{\alpha}} - E^{-\hat{\alpha}}). \quad (3.75)$$

Here by $-\hat{\alpha}$ we mean $(-\alpha, -n)$ for $\hat{\alpha} = (\alpha, n)$ and $(\mu, -n)$ for $\hat{\alpha} = (\mu, n)$. It turns out that this basis is orthogonal with respect to the form $(\cdot, \cdot)_{\mathcal{G}}$ defined in (7.1). The dual basis to (3.75) will be denoted as $t_\mu, b_{\hat{\alpha}}, c_{\hat{\alpha}}$, $\hat{\alpha} \in \hat{\Phi}_+$.

Now we can finally write down the basis of $\hat{\mathcal{G}} = \widehat{L\mathcal{G}_0}$ alluded above; it reads

$$\hat{T}^\infty, \iota(T^\mu), \iota(B^{\hat{\alpha}}), \iota(C^{\hat{\alpha}}), \quad \hat{\alpha} \in \hat{\Phi}_+. \quad (3.76)$$

The dual basis is

$$\hat{t}_\infty, \pi^*(t^\mu), \pi^*(b^{\hat{\alpha}}), \pi^*(c^{\hat{\alpha}}), \quad \hat{\alpha} \in \hat{\Phi}_+, \quad (3.77)$$

where the map $\pi^* : \mathcal{G}^* \rightarrow \hat{\mathcal{G}}^*$ is induced by the exact sequence (2.2).

By using the general formula (7.14) and the explicit form (7.16) of the co-cycle, there is no problem to write down all commutation relations among the generators of the basis (3.76). Here we shall write down only the commutators relevant for further discussion, or, in other words, only the commutators which are not annihilated by all elements of $Span(\hat{t}_\infty, \pi^*(t_\mu))$. Thus we have for every $\hat{\alpha} \in \hat{\Phi}_+$

$$[\iota(B^{\hat{\alpha}}), \iota(C^{\hat{\alpha}})] = -i\hat{\alpha}^\vee, \quad (3.78)$$

where $\hat{\alpha}^\vee$ is the so-called affine coroot. It is given explicitly as follows

$$-i\hat{\alpha}^\vee = \iota(-i\alpha^\vee) - \frac{2n}{|\alpha|^2}\hat{T}^\infty, \quad \hat{\alpha} = (\alpha, n), \quad -i\hat{\alpha}^\vee = -n\hat{T}^\infty, \quad \hat{\alpha} = (\mu, n). \quad (3.79)$$

Now we are ready to study the symplectic form $\hat{\omega}_L = d\hat{\theta}_L$ on the reduced model space \hat{M}_L . In what follows, we shall suppress the subscript L on the coordinates $(\hat{k}_L, \hat{\phi}_L)$ of the model space. First of all, $\hat{\omega}_L$ can be written as

$$\begin{aligned} \hat{\omega}_L &= \langle d\hat{\phi} \hat{\lrcorner} \hat{k}^{-1}d\hat{k} \rangle - \langle \hat{\phi}, \hat{k}^{-1}d\hat{k} \wedge \hat{k}^{-1}d\hat{k} \rangle = \\ &= da^\infty \wedge L_{\hat{k}^{-1}}^* \hat{t}_\infty + d(a^\infty a^\mu) \wedge L_{\hat{k}^{-1}}^* \pi^*(t_\mu) + \\ &\quad + \sum_{\hat{\alpha} \in \hat{\Phi}_+} \langle \hat{\phi}, i\hat{\alpha}^\vee \rangle L_{\hat{k}^{-1}}^* \pi^*(b_{\hat{\alpha}}) \wedge L_{\hat{k}^{-1}}^* \pi^*(c_{\hat{\alpha}}). \end{aligned} \quad (3.80)$$

Here we have set

$$\hat{\phi} = a^\infty \hat{t}_\infty + a^\infty a^\mu \pi^*(t_\mu) \quad (3.81)$$

and used

$$\begin{aligned} \hat{k}^{-1}d\hat{k} &= L_{\hat{k}^{-1}}^* \hat{t}_\infty \otimes \hat{T}^\infty + L_{\hat{k}^{-1}}^* \pi^*(t_\mu) \otimes \iota(T^\mu) + \\ &\quad + L_{\hat{k}^{-1}}^* \pi^*(b_{\hat{\alpha}}) \otimes \iota(B^{\hat{\alpha}}) + L_{\hat{k}^{-1}}^* \pi^*(c_{\hat{\alpha}}) \otimes \iota(C^{\hat{\alpha}}). \end{aligned} \quad (3.82)$$

Note that the normalization is chosen in the way that a^μ 's parametrize the alcove \mathcal{A}_+^1 , i.e. $a^\mu t_\mu \in \mathcal{A}_+^1$. We have

$$\langle \hat{\phi}, i\hat{\alpha}^\vee \rangle = \frac{2a^\infty}{|\alpha|^2} (n + \langle \alpha, H^\mu \rangle a^\mu), \quad \hat{\alpha} = (\alpha, n); \quad (3.83)$$

$$\langle \hat{\phi}, i\hat{\alpha}^\vee \rangle = na^\infty, \quad \hat{\alpha} = (\mu, n). \quad (3.84)$$

This follows from the well-known fact

$$\alpha^\vee = \frac{2}{|\alpha|^2} \langle \alpha, H^\mu \rangle H^\mu. \quad (3.85)$$

Theorem 3.7: The symplectic reduction of $\hat{\omega}_L$, induced by setting the moment map $\langle \hat{\beta}_L(\hat{k}\hat{\phi}), \hat{T}^\infty \rangle$ equal to some real number κ , gives the chiral WZW symplectic form ω_L^{WZ} on the WZW model space $M_L^{WZ} = G \times \mathcal{A}_+^1 = LG_0 \times \mathcal{A}_+^1$.

Proof: First we observe that $\langle \hat{\beta}_L(\hat{k}\hat{\phi}), \hat{T}^\infty \rangle = a^\infty$. The form $\hat{\omega}_L$ restricted to the surface $a^\infty = \kappa$ becomes

$$\begin{aligned} \hat{\omega}_L|_{\hat{a}^\infty=\kappa} &= \kappa da^\mu \wedge \pi^* L_{\pi(\hat{k})^{-1}}^* t_\mu + \\ &+ \kappa \sum_{\hat{\alpha}=(\alpha,n) \in \hat{\Phi}_+} \frac{2}{|\alpha|^2} (n + \langle \alpha, H^\mu \rangle a^\mu) \pi^* L_{\pi(\hat{k})^{-1}}^* b_{\hat{\alpha}} \wedge \pi^* L_{\pi(\hat{k})^{-1}}^* c_{\hat{\alpha}} + \\ &+ \kappa \sum_{\hat{\alpha}=(\mu,n) \in \hat{\Phi}_+} n \pi^* L_{\pi(\hat{k})^{-1}}^* b_{\hat{\alpha}} \wedge \pi^* L_{\pi(\hat{k})^{-1}}^* c_{\hat{\alpha}}. \end{aligned} \quad (3.86)$$

In deriving this formula, we have used (3.83) and (3.84) and also the fact that π is the group homomorphism, which implies that $\pi^* L_{\pi(\hat{k})^{-1}}^* = L_{\hat{k}^{-1}}^* \pi^*$.

Since we know that the moment map $\langle \hat{\beta}_L(\hat{k}\hat{a}), \hat{T}^\infty \rangle = a^\infty$ generates the central circle action, we conclude immediately that the kernel of the form $\hat{\omega}_L$ restricted to $a^\infty = \kappa$ is spanned by the vectors $L_{\hat{k}*} \hat{T}^\infty$. This can be seen also directly from the formula (3.86) since the central circle does not act on the coordinates a^μ, a^∞ of the reduced affine model space \hat{M}_L . The restricted form (3.86) is therefore pullback of some two-form ω'_L on the manifold $M_L = G \times \mathcal{A}_+^1$ by the map $\pi : (\hat{k}, a^\mu) \rightarrow (\pi(\hat{k}), a^\mu)$. It remains to find this two-form ω'_L on M_L . The first term in (3.86) can be rewritten as

$$\kappa da^\mu \wedge \pi^* L_{\pi(\hat{k})^{-1}}^* t_\mu = \kappa \pi^* \left(da^\mu \wedge \langle t_\mu, k^{-1} dk \rangle \right). \quad (3.87)$$

Then we have

$$\begin{aligned} \kappa \sum_{\hat{\alpha}=(\alpha,n) \in \hat{\Phi}_+} \frac{2}{|\alpha|^2} \langle \alpha, H^\mu \rangle a^\mu \pi^* L_{\pi(\hat{k})^{-1}}^* b_{\hat{\alpha}} \wedge \pi^* L_{\pi(\hat{k})^{-1}}^* c_{\hat{\alpha}} &= \\ &= -\kappa \pi^* \left(\langle a^\mu t_\mu, k^{-1} dk \wedge k^{-1} dk \rangle \right). \end{aligned} \quad (3.88)$$

Here we have used the commutation relations in the Lie algebra $L\mathcal{G}_0$:

$$[B^{\hat{\alpha}}, C^{\hat{\alpha}}] = -i\alpha^\vee, \quad \hat{\alpha} = (\alpha, n); \quad (3.89)$$

$$[B^{\hat{\alpha}}, C^{\hat{\alpha}}] = 0, \quad \hat{\alpha} = (\mu, n). \quad (3.90)$$

By using the same commutation relations and the cocycle formula (7.16), we directly find that the remaining term proportional to κ is in fact equal to $-\frac{\kappa}{2}\pi^*(k^{-1}dk \frown \partial_\sigma(k^{-1}dk))_{\mathcal{G}}$. Putting all together

$$\omega'_L = \kappa da^\mu \wedge \langle t_\mu, k^{-1}dk \rangle - \kappa \langle a^\mu t_\mu, k^{-1}dk \wedge k^{-1}dk \rangle - \frac{\kappa}{2}(k^{-1}dk \wedge \partial_\sigma(k^{-1}dk))_{\mathcal{G}}. \quad (3.91)$$

Now we make a comparison with the formula (4.5) of [20] to conclude that, up to the (2π) normalization (cf. Section 2.2.3), our ω'_L is indeed the symplectic form ω_L^{WZ} of the chiral WZW model.

The theorem is proved. #

We conclude this paragraph by writing the formula for the (doubly) reduced Hamiltonian on the WZW model space M_L^{WZ} . It can be read off from the formula (3.72) :

$$H_L^{WZ} = -\frac{1}{2\kappa}(\phi_L, \phi_L)_{\mathcal{G}^*} - \langle \phi_L, k_L^{-1}\partial k_L \rangle - \frac{\kappa}{2}(k_L^{-1}\partial k_L, k_L^{-1}\partial k_L)_{\mathcal{G}}, \quad (3.92)$$

where $\phi_L = \kappa a^\mu t_\mu$. This coincides with the Sugawara Hamiltonian of the chiral WZW model as we shall see in Section 3.2.6. Having obtained the correct symplectic form and Hamiltonian, we have indeed produced the standard chiral WZW theory by the two-step chiral symplectic reduction from the chiral master model (1.2). The fact that the full left-right WZW model can be obtained by glueing two chiral WZW theories was explained e.g. in [20] and we shall not repeat this argument here.

3.2.5 Affine dynamical r -matrix

Our next task is to prepare the land for the quasitriangular generalization described later on in this paper. For this we have to invert the chiral WZW symplectic form ω_L^{WZ} . We shall write it as follows

$$\omega_L^{WZ} = \kappa da^\mu \wedge L_{k^{-1}}^* t_\mu + \sum_{\hat{\alpha} \in \hat{\Phi}_+} \langle \hat{\phi}_\kappa, i\hat{\alpha}^\vee \rangle L_{k^{-1}}^* b_{\hat{\alpha}} \wedge L_{k^{-1}}^* c_{\hat{\alpha}}, \quad (3.93)$$

where

$$\hat{\phi}_\kappa = \kappa \hat{t}_\infty + \kappa a^\mu \pi^*(t_\mu). \quad (3.94)$$

From here we find immediately the corresponding Poisson bivector Π_L^{WZ} :

$$\Pi_L^{WZ}(k, a^\mu) = -\frac{\partial}{\kappa \partial a^\mu} \wedge L_{k*} T^\mu - \sum_{\hat{\alpha} \in \hat{\Phi}_+} \frac{1}{\langle \hat{\phi}_\kappa, i\hat{\alpha}^\vee \rangle} L_{k*} B^{\hat{\alpha}} \wedge L_{k*} C^{\hat{\alpha}}. \quad (3.95)$$

Let us calculate the Poisson bracket of certain special functions of the variables $(k, a^\mu) \in G \times \mathcal{A}_+^1$. These functions are simply the matrix elements in some representation of LG_0 . The bivector formula (3.95) then immediately implies the following Poisson brackets

$$\{k \otimes k\}_{WZ} = (k \otimes k) \hat{r}_0(\hat{\phi}_\kappa), \quad (3.96)$$

$$\{k, a^\mu\}_{WZ} = k T^\mu, \quad \{a^\mu, a^\nu\}_{WZ} = 0, \quad (3.97)$$

where

$$\hat{r}_0(\hat{\phi}_\kappa) = \sum_{\hat{\alpha} \in \hat{\Phi}} \frac{i}{\langle \hat{\phi}_\kappa, i\hat{\alpha}^\vee \rangle} E^{\hat{\alpha}} \otimes E^{-\hat{\alpha}}. \quad (3.98)$$

The brackets (3.96) and (3.97) characterize completely the Poisson structure on the WZW model space M_L^{WZ} . Note also that the summation in (3.98) is not restricted only to the positive roots.

The fundamental braiding relation (3.96) can be rewritten in the representation corresponding to the pointwise action of the loop group element on the G_0 -representation space V_0 . In other words, consider a function on LG_0 of the form $\rho_{ij}^{\sigma'}(k)$, where ρ is a matrix representation of the group G_0 and σ' is some point on the loop. In words, this function is defined as follows: take an element $(k, a^\mu) \in M_L^{WZ}$, forget about a^μ , consider the element $k(\sigma')$ of the group G_0 obtained by evaluation of k at the point σ' and finally take the matrix element ij of the element $k(\sigma')$ in the G_0 -representation ρ . If we recall the definition (3.75) of $B^{\hat{\alpha}}$ and $C^{\hat{\alpha}}$ in terms of $E^{\hat{\alpha}}$ and the definition (3.74) of $E^{\hat{\alpha}}$ in terms of E^α , H^μ and $e^{in\sigma}$, we can directly derive from (3.98)

$$\{k(\sigma) \otimes k(\sigma')\}_{WZ} = (k(\sigma) \otimes k(\sigma')) \hat{r}_0(\hat{\phi}_\kappa, \sigma - \sigma'), \quad (3.99)$$

where the affine dynamical r -matrix is denoted as $\hat{r}_0(\hat{\phi}_\kappa, \sigma - \sigma')$ and defined as

$$\hat{r}_0(\hat{\phi}_\kappa, \sigma - \sigma') = i \sum_{\alpha \in \Phi, n \in \mathbf{Z}} \frac{|\alpha|^2}{2} \frac{1}{\kappa(n + \langle \alpha, H^\mu \rangle a^\mu)} E^\alpha \otimes E^{-\alpha} \exp in(\sigma - \sigma')$$

$$+ i \sum_{\mu, n \in \mathbf{Z}, n \neq 0} \frac{1}{n\kappa} H^\mu \otimes H^\mu \exp in(\sigma - \sigma'). \quad (3.100)$$

It is important to note that the summation goes over all roots $\alpha \in \Phi$, not only over the positive ones. From (3.97), we can also derive the following bracket

$$\{k(\sigma), a^\mu\}_{WZ} = \frac{1}{\kappa} k(\sigma) T^\mu. \quad (3.101)$$

It is simple to sum up the Fourier series in (3.100). The result is

$$\begin{aligned} \hat{r}_0(\hat{\phi}_\kappa, \sigma - \sigma') &= \frac{1}{\kappa} \text{Per}(\sigma - \sigma' - \pi) \sum_{\mu} H^\mu \otimes H^\mu + \\ &+ \sum_{\alpha \in \Phi} \frac{|\alpha|^2}{2\kappa} \frac{2\pi}{e^{-2\pi i a^\mu \langle \alpha, H^\mu \rangle} - 1} \text{Per}(e^{-ia^\mu \langle \alpha, H^\mu \rangle} (\sigma - \sigma')) E^\alpha \otimes E^{-\alpha}, \end{aligned} \quad (3.102)$$

where the notation $\text{Per}(f(\sigma))$ means the function of σ periodic with the period 2π and defined as $f(\sigma)$ for $\sigma \in [0, 2\pi]$.

3.2.6 Vertex-IRF transformation and braiding relation

The formula of the type (3.96) appears in the WZW literature [3, 19, 10, 14, 15] under the name of the exchange (braiding) relation. The reader might have noticed however, that our formula (3.99) does not at all resemble e.g. the braiding relation (26,31) of the reference [14]. The reason is that we have used different coordinates on the WZW model space M_L^{WZ} . In order to establish the equivalence of our approach with that of [14], we must perform the so-called classical vertex-IRF transformation (the terminology is borrowed from [20]).

Consider a map $\sigma \rightarrow m(\sigma) \in G_0$ defined as

$$m(\sigma) = k(\sigma) \exp(a^\mu \Upsilon(t_\mu) \sigma), \quad (3.103)$$

where (k, a^μ) are the old coordinates of the WZW model space M_L^{WZ} . Now we introduce the new set of "monodromic" coordinates $m(\sigma)$. The name is motivated by the fact that $m(\sigma)$ is no longer a single-valued function but it develops a monodromy upon going around the circle of sigmas. This monodromy is encoded in the variables a^μ , hence $m(\sigma)$ encodes the information about both k and a^μ .

We wish to calculate the exchange relation (3.99) in terms of the variables m . We shall use the following obvious matrix relation

$$\begin{aligned} \{AB \otimes CD\} &= (A \otimes 1)\{B \otimes C\}(1 \otimes D) + (A \otimes C)\{B \otimes D\} + \\ &+ \{A \otimes C\}(B \otimes D) + (1 \otimes C)\{A \otimes D\}(B \otimes 1) \end{aligned} \quad (3.104)$$

and write

$$\begin{aligned} \{m(\sigma) \otimes m(\sigma')\}_{WZ} &= \{k(\sigma)e^{a^\mu \Upsilon(t_\mu)\sigma} \otimes k(\sigma')e^{a^\mu \Upsilon(t_\mu)\sigma'}\}_{WZ} = \\ &= (m(\sigma) \otimes m(\sigma')) \left[-\frac{\sigma - \sigma'}{\kappa} (H^\mu \otimes H^\mu) + \right. \\ &+ \left. (e^{-a^\mu \Upsilon(t_\mu)\sigma} \otimes e^{-a^\mu \Upsilon(t_\mu)\sigma'}) \hat{r}_0(\hat{\phi}_\kappa, \sigma - \sigma') (e^{a^\mu \Upsilon(t_\mu)\sigma} \otimes e^{a^\mu \Upsilon(t_\mu)\sigma'}) \right] \equiv \\ &\equiv (m(\sigma) \otimes m(\sigma')) B_0(\hat{\phi}_\kappa, \sigma - \sigma'). \end{aligned} \quad (3.105)$$

We shall call $B_0(\hat{\phi}_\kappa, \sigma - \sigma')$ the quasiclassical braiding matrix. It is important to note that the argument σ of the braiding matrix $B_0(\hat{\phi}_\kappa, \sigma)$ is the element of \mathbf{R} and not of S^1 . This is related to the fact that the monodromic coordinate $m(\sigma)$ on M_L is multi-valued from the point of view of S^1 . Considering σ as an element of \mathbf{R} makes the quantity $m(\sigma)$ single-valued and the Poisson bracket (3.105) well-defined.

Now combining (3.102) and (3.105), we immediately arrive at (cf. [14])

$$B_0(\hat{\phi}_\kappa, \sigma) = -\frac{\pi}{\kappa} \left[\eta(\sigma)(H^\mu \otimes H^\mu) - i \sum_\alpha \frac{|\alpha|^2}{2} \frac{\exp(i\pi\eta(\sigma)\langle\alpha, H^\mu\rangle a^\mu)}{\sin(\pi\langle\alpha, H^\mu\rangle a^\mu)} E^\alpha \otimes E^{-\alpha} \right]. \quad (3.106)$$

Here $\eta(\sigma)$ is the function defined by

$$\eta(\sigma) = 2\left[\frac{\sigma}{2\pi}\right] + 1, \quad (3.107)$$

where $[\sigma/2\pi]$ is the largest integer less than or equal to $\frac{\sigma}{2\pi}$.

It turns out that important dynamical variables can be particularly simply expressed in terms of the new variables $m(\sigma)$. Before showing this, some discussion is needed about the moment maps generating the left \hat{G} -action on \hat{M}_L and about their behaviour under the symplectic reduction leading from the reduced affine model space \hat{M}_L to the WZW model space M_L^{WZ} .

We already know that the moment maps $\hat{\beta}_L(\hat{k}\phi) = \widehat{Coad}_{\hat{k}}\hat{\phi}$ generate (via the Poisson bracket on \hat{M}_L) the left action of the group \hat{G} on \hat{M}_L . In particular, the moment $\langle \hat{\beta}_L, \hat{T}^\infty \rangle$ generates the central circle action on $\hat{M}_L = \hat{G} \times \hat{\mathcal{A}}_+$. This means that

$$\{\langle \hat{\beta}_L, \hat{T}^\infty \rangle, \langle \hat{\beta}_L, \iota(x) \rangle\}_{\hat{M}_L} = 0, \quad x \in \mathcal{G}. \quad (3.108)$$

It then follows that

$$\langle \hat{\beta}_L(\hat{k}\phi), \iota(x) \rangle = \langle \hat{\beta}_L(e^{s\hat{T}^\infty} \hat{k}\phi), \iota(x) \rangle, \quad x \in \mathcal{G}; \quad (3.109)$$

or, in other words, $\langle \hat{\beta}_L, \iota(x) \rangle$ is invariant function with respect to the central circle action. As such, it gives rise to some function on the space of the central circle orbits located at the submanifold $a^\infty = \kappa$ of the affine model space \hat{M}_L . The latter space of orbits is nothing but the reduced model space M_L^{WZ} hence we conclude that $\langle \hat{\beta}_L, \iota(x) \rangle$ can be interpreted as the honest function on M_L^{WZ} . We denote it as $j_L^x(k, a^\mu)$.

Actually, the functions $j_L^x(k, a^\mu)$ are the "important dynamical variables" mentioned above. In fact they are nothing but the generators of the chiral current algebra. To see this, we calculate their Poisson brackets $\{j_L^x, j_L^y\}_{WZ}$. The computation follows the general procedure of the symplectic reduction at the level of the Poisson brackets as described in the Appendix 7.3.

Consider a pair of functions ϕ_i , $i = 1, 2$ on M_L^{WZ} . We wish to calculate their reduced Poisson bracket $\{\phi_1, \phi_2\}_{WZ}$. In our particular situation, the procedure works as follows: define two functions $\hat{\phi}_i$ on \hat{M}_L that fulfil

$$\hat{\phi}_i(\hat{k}, a^\mu, a^\infty = \kappa) = \phi_i(\pi(\hat{k}), a^\mu), \quad \hat{k} \in \widehat{LG}_0. \quad (3.110)$$

Calculate then the Poisson bracket $\{\hat{\phi}_1, \hat{\phi}_2\}_{\hat{M}_L}$ on \hat{M}_L . It verifies

$$\{a^\infty, \{\hat{\phi}_1, \hat{\phi}_2\}_{\hat{M}_L}\}_{\hat{M}_L} = 0 \quad (3.111)$$

as the simple consequence of the Jacobi identity and the central circle invariance of $\hat{\phi}_i$. This means that it exists a function on M_L^{WZ} denoted suggestively as $\{\phi_1, \phi_2\}_{WZ}$ which verifies

$$\{\hat{\phi}_1, \hat{\phi}_2\}_{\hat{M}_L}(\hat{k}, a^\mu, a^\infty = \kappa) = \{\phi_1, \phi_2\}_{WZ}(\pi(\hat{k}), a^\mu). \quad (3.112)$$

Needless to say, the function $\{\phi_1, \phi_2\}_{WZ}$ is the seeked reduced Poisson bracket.

Now the function $\langle \hat{\beta}_L, \iota(x) \rangle$ on \hat{M}_L plays the role of $\hat{\phi}_1$ for the function $\phi_1 = j_L^x$ on M_L^{WZ} . But we know from (3.64) and (2.7) that

$$\{\langle \hat{\beta}_L, \iota(x) \rangle, \langle \hat{\beta}_L, \iota(y) \rangle\}_{\hat{M}_L} = \langle \hat{\beta}_L, \iota([x, y]) \rangle + \langle \hat{\beta}_L, \hat{T}^\infty \rangle \rho(x, y). \quad (3.113)$$

Using the fact that $\langle \hat{\beta}_L, \hat{T}^\infty \rangle = a^\infty$ and the relation (3.112), we obtain immediately

$$\{j_L^x, j_L^y\}_{WZ} = j_L^{[x, y]} + \kappa \rho(x, y). \quad (3.114)$$

This is nothing but the basic relation defining the chiral current algebra.

Let us calculate the currents j_L^x as explicit functions of k, a^μ . By using the formulas (3.38) and (2.24), we infer

$$\begin{aligned} \langle \hat{\beta}_L(\hat{k}\hat{\phi}), \iota(x) \rangle &= \langle \widehat{Coad_{\hat{k}}}(a^\infty \hat{t}_\infty + a^\infty a^\mu \pi^*(t_\mu)), \iota(x) \rangle = \\ &= \langle \pi^*(Coad_k(a^\infty a^\mu t_\mu) + a^\infty \Upsilon^{-1}(\partial_\sigma k k^{-1})), \iota(x) \rangle, \end{aligned} \quad (3.115)$$

where $k = \pi(\hat{k})$. Thus for the currents we obtain

$$j_L^x(k, a^\mu) = \kappa(a^\mu k \Upsilon(t_\mu) k^{-1} + \partial_\sigma k k^{-1}, x)_{\mathcal{G}}. \quad (3.116)$$

This expression look quite complicated but it drastically simplifies in the monodromic variables $m(\sigma)$:

$$j_L^x = \kappa(\partial_\sigma m m^{-1}, x)_{\mathcal{G}}. \quad (3.117)$$

Moreover, also the Hamiltonian H_L^{WZ} given by (3.92) simplifies considerably:

$$H_L^{WZ} = -\frac{1}{2\kappa}(\kappa \partial_\sigma m m^{-1}, \kappa \partial_\sigma m m^{-1})_{\mathcal{G}}. \quad (3.118)$$

This is the Sugawara formula expressing the chiral WZW Hamiltonian solely in terms of the Kac-Moody currents. The reader should not be confused by the minus sign. It is related to the negative definiteness of our bilinear form $(\cdot, \cdot)_{\mathcal{G}}$.

The monodromic variables are generally used in the study of the standard WZW model. It turns out, however, that viewing things from the points of view of the variables k, a^μ will be more insightful for seeking the quasitriangular generalization of the story.

Another important Poisson bracket is the following one

$$\{m(\sigma), (\kappa \partial_\sigma m(\sigma') m^{-1}(\sigma'), x_0)_{\mathcal{G}_0}\}_{WZ} = x_0 m(\sigma) \delta(\sigma - \sigma'), \quad x_0 \in \mathcal{G}_0. \quad (3.119)$$

It is the direct consequence of the braiding relation (3.105) and it expresses the simple fact, that even after the symplectic reduction the current $\kappa \partial_\sigma m m^{-1}$ continues to generate the left action of the centrally extended loop group on the WZW model space M_L^{WZ} . However, the action of the generator of the central circle is now trivial. Upon the quantization, the bracket (3.119) means that $m(\sigma)$ is the Kac-Moody primary field.

Finally, as the illustration of the suitability of the monodromic variables, we give the explicit expression for the solution of the classical field equations of the chiral WZW model. It turns out, that the time evolution of the coordinates $m(\sigma)$ is given by the following simple formula

$$[m(\sigma)](\tau) = m(\sigma - \tau). \quad (3.120)$$

This fact can be derived from the braiding relation (3.105) and from the Sugawara formula (3.118) (because those two ingredients anyway characterize completely the structure of the standard chiral WZW model). Here we shall offer another derivation based on the solution of the second floor chiral master model (1.2). Indeed, varying the action (1.2), we find in full analogy with the finite-dimensional calculation of Section 3.1.3 that the classical solution of the second floor chiral model

$$\tilde{k}(\tau) = \tilde{k}_0 \exp\left(\frac{-\tilde{\Upsilon}(\tilde{\phi}_0)}{\kappa} \tau\right). \quad (3.121)$$

Since we have performed the reduction with respect to the *left* action of the group \mathbf{R}_S , we have to cast $\tilde{k}(\tau)$ as

$$\tilde{k}(\tau) = e^{s(\tau)\tilde{T}^0} \hat{k}(\tau), \quad (3.122)$$

and suppress $e^{s(\tau)\tilde{T}^0} \in \mathbf{R}_S$. This corresponds to the first step of the reduction. The result is

$$\hat{k}(\tau) = \left(e^{-\frac{a^\infty \tau}{\kappa} \tilde{T}^0} \hat{k}_0 e^{\frac{a^\infty \tau}{\kappa} \tilde{T}^0}\right) e^{\frac{1}{\kappa} (a^0 \hat{T}^\infty - a^\infty a^\mu \Upsilon(t_\mu)) \tau}, \quad (3.123)$$

The reduction $a^\infty = \kappa$ to the ground floor gives

$$k(\tau) = \pi(\hat{k}(\tau)) = k_0(\sigma - \tau) e^{-a^\mu \Upsilon(t_\mu) \tau}, \quad (3.124)$$

where $\pi(\hat{k}_0) \equiv k_0$. Combining the definition (3.103) of the monodromic variables with the last formula (3.124), we arrive directly at the desired relation (3.120).

Chapter 4

Universal quasitriangular WZW model

In two preceding sections, we have described two important dynamical systems. The first one - the geodesical model - was constructed for any Lie group G possessing an invariant symmetric non-degenerate bilinear form on the Lie algebra \mathcal{G} of G . The construction of the second one - the WZW model - necessitated moreover an existence of the central biextension \tilde{G} of G . The symplectic structures of both these models have been either identified to or derived from the canonical symplectic structure of the cotangent bundle T^*G (for the geodesical model) and of $T^*\tilde{G}$ (for the WZW model).

In this section, we are going to show that one can generalize both geodesical and WZW models mentioned above. A recipe how to do this consists in a crucial observation that T^*G is the so called Heisenberg double of the group G . The latter is a certain group equipped with an additional structure that we shall describe in what follows. There may exist many different Heisenberg doubles of a given group G ; for us it is important that the geodesical model and the WZW one can be constructed by using only those properties of T^*G that are shared also by all others Heisenberg doubles of G . In particular, given a nontrivial Heisenberg double of the centrally biextended group \tilde{G} , we can define an associated WZW-like model. We shall refer to the latter as to the universal quasitriangular WZW model.

In order to keep this paper as selfcontained as possible, we are going to give here a quick review of theory of Poisson-Lie groups and of related various doubles of Lie groups. The reader may find somewhat inconvenient to bur-

den the text also with demonstrations of the relevant standard propositions. However, these demonstrations involve many important facts, technical skills and computational tools that may facilitate the understanding of the present article.

4.1 Poisson-Lie primer

4.1.1 The Drinfeld double

A Poisson bracket on a Lie group manifold G that is compatible with the group multiplication law is called the **Poisson-Lie** bracket. Denote $\Delta : Fun(G) \rightarrow Fun(G) \otimes Fun(G)$ the standard coproduct defined as

$$(\Delta F)(g, g') = F(gg'), \quad g, g' \in G, \quad F \in Fun(G). \quad (4.1)$$

Then the condition of compatibility reads

$$\{\Delta F_1, \Delta F_2\}_{G \times G} = \Delta\{F_1, F_2\}_G. \quad (4.2)$$

Here $\{.,.\}_{G \times G}$ is the direct product Poisson bracket on $G \times G$ characterized by the condition

$$\begin{aligned} & \{F_1(x)G_1(y), F_2(x)G_2(y)\}_{G \times G} = \\ & = \{F_1(x), F_2(x)\}_G G_1(y)G_2(y) + F_1(x)F_2(x)\{G_1(y), G_2(y)\}_G, \end{aligned} \quad (4.3)$$

where x and y are coordinates on the first and second copy of G respectively. Note that upon the quantization of the algebra of functions on the group manifold, we obtain the so called quantum group. The quantum version of the Poisson-Lie condition (4.2) then becomes the usual statement in the theory of Hopf algebras saying that the coproduct is the algebra homomorphism.

On a given group G one may have several inequivalent Poisson-Lie brackets. In this paper, we shall always be concerned with one privileged way of constructing the Poisson-Lie structures on G that uses the concept of the Heisenberg double of G . Before defining this notion, we have to recall respectively the definitions of the Manin and Drinfeld doubles.

So the **Manin** double of a Lie group G is any Lie group D whose dimension is twice as big as that of G and that fulfils two other conditions:

- 1) The double D contains G as its subgroup;

2) The Lie algebra \mathcal{D} of the double D is equipped with an invariant symmetric nondegenerate bilinear form $(\cdot, \cdot)_{\mathcal{D}}$ such that the Lie algebra \mathcal{G} of G is isotropic; i.e. $(\mathcal{G}, \mathcal{G})_{\mathcal{D}} = 0$.

Suppose now that the same group D equipped with the form $(\cdot, \cdot)_{\mathcal{D}}$ on \mathcal{D} is the Manin double of two Lie groups G and B . We say that D is the **Drinfeld** double of G (and of B), if the Lie algebras \mathcal{G} and \mathcal{B} of G and B linearly generate all the Lie algebra \mathcal{D} of D . In other words:

$$\mathcal{D} = \mathcal{G} + \mathcal{B}, \quad (\mathcal{G}, \mathcal{G})_{\mathcal{D}} = 0, \quad (\mathcal{B}, \mathcal{B})_{\mathcal{D}} = 0. \quad (4.4)$$

It is very important to realize that $\mathcal{G} + \mathcal{B}$ means here the direct sum of two vector spaces and not the direct sum of two Lie algebras. The Lie bracket on \mathcal{D} can be conveniently encoded in terms of the Lie brackets on \mathcal{G} and \mathcal{B} and of the form $(\cdot, \cdot)_{\mathcal{D}}$ as follows

$$[X + \alpha, Y + \beta]_{\mathcal{D}} = [X, Y]_{\mathcal{G}} + [\alpha, \beta]_{\mathcal{B}} + \text{Coad}_X \beta - \text{Coad}_Y \alpha + \text{Coad}_{\alpha} Y - \text{Coad}_{\beta} X. \quad (4.5)$$

Here $X, Y \in \mathcal{G}$ and $\alpha, \beta \in \mathcal{B}$. As far as the coadjoint action is concerned, the elements α, β are viewed as the elements of \mathcal{G}^* by the prescription

$$\langle \alpha, X \rangle = (\alpha, X)_{\mathcal{D}}, \quad (4.6)$$

and similarly X, Y are viewed as the elements of \mathcal{B}^* under the coadjoint action of \mathcal{B} .

4.1.2 The Heisenberg double

The group multiplication law in the Drinfeld double D induces smooth maps $\mathcal{M}_L : G \times B \rightarrow D$ and $\mathcal{M}_R : B \times G \rightarrow D$ given by

$$\mathcal{M}_L(g, b) = gb, \quad \mathcal{M}_R(b, g) = bg, \quad g \in G, \quad b \in B. \quad (4.7)$$

The crucial fact underlying all this article can be formulated as the following theorem

Theorem 4.1 : If the maps $\mathcal{M}_{L,R}$ defined above are bijective then D is a symplectic manifold and the following expression defines the symplectic form

$$\omega = \frac{1}{2}(b_L^* \lambda_B \frown g_R^* \rho_G)_{\mathcal{D}} + \frac{1}{2}(b_R^* \rho_B \frown g_L^* \lambda_G)_{\mathcal{D}}. \quad (4.8)$$

The corresponding Poisson bracket reads

$$\{\phi, \psi\}_D = \frac{1}{2}(\nabla^L \phi, \mathcal{R}^* \nabla^L \psi)_{\mathcal{D}^*} + \frac{1}{2}(\nabla^R \phi, \mathcal{R}^* \nabla^R \psi)_{\mathcal{D}^*}. \quad (4.9)$$

Remark: The Poisson bracket (4.9) was introduced by Semenov-Tian-Shansky in [44]. If the maps $\mathcal{M}_{L,R}$ are not bijective then (4.9) still defines a Poisson bracket and the symplectic leaves of the corresponding Poisson structure were described in [2]. The bijectiveness of $\mathcal{M}_{L,R}$ is often referred to as the property that the group D is smoothly globally decomposable as $D = GB$ and $D = BG$.

Let us explain the symbols appearing in (4.8): The maps $b_L : D \rightarrow B$ and $g_R : D \rightarrow G$ are induced by the decomposition $D = BG$ and $g_L : D \rightarrow G$ and $b_R : D \rightarrow B$ by $D = GB$. The expression λ_G (ρ_G) denotes the left(right)invariant \mathcal{G} -valued Maurer-Cartan form on the group G . Recall that

$$\lambda_G(X_g) = L_{g^{-1}*}X_g, \quad \rho_G(X_g) = R_{g^{-1}*}X_g, \quad X_g \in T_g G. \quad (4.10)$$

Note that the forms λ_G and ρ_G are often written also as

$$\lambda_G = g^{-1}dg, \quad \rho_G = dgg^{-1}. \quad (4.11)$$

The notation used in (4.9) is as follows: $(.,.)_{\mathcal{D}^*}$ is the bilinear form on the dual of \mathcal{D} induced by the (nondegenerate) bilinear form $(.,.)_{\mathcal{D}}$ and $\mathcal{R}^* : \mathcal{D}^* \rightarrow \mathcal{D}^*$ is the map dual to $\mathcal{R} : \mathcal{D} \rightarrow \mathcal{D}$. The latter is given by

$$\mathcal{R} = Pr_{\mathcal{B}} - Pr_{\mathcal{G}}, \quad (4.12)$$

where $Pr_{\mathcal{G}}$ ($Pr_{\mathcal{B}}$) is the projector on \mathcal{G} (\mathcal{B}) with the kernel \mathcal{B} (\mathcal{G}). Clearly, the decomposition $\mathcal{D} = \mathcal{G} + \mathcal{B}$ induces the corresponding decomposition of the dual $\mathcal{D}^* = \mathcal{G}^* + \mathcal{B}^*$ and

$$\mathcal{R}^* = Pr_{\mathcal{B}^*} - Pr_{\mathcal{G}^*}. \quad (4.13)$$

Recall also the definitions (cf. (7.26),(7.27) of the differential operators ∇^L, ∇^R on D :

$$\nabla^L : Fun(D) \rightarrow Fun(D) \otimes \mathcal{D}^*; \quad \nabla^R : Fun(D) \rightarrow Fun(D) \otimes \mathcal{D}^*; \quad (4.14)$$

$$\langle \nabla^L \phi, \alpha \rangle(K) = \left(\frac{d}{ds} \right)_{s=0} \phi(e^{s\alpha} K), \quad \langle \nabla^R \phi, \alpha \rangle(K) = \left(\frac{d}{ds} \right)_{s=0} \phi(K e^{s\alpha}). \quad (4.15)$$

Here $\alpha \in \mathcal{D}$, $K \in D$ and $\phi \in \text{Fun}(D)$.

It is useful to write the bracket (4.9) in some basis $T^i, t_i; i = 1, \dots, \dim G$ of \mathcal{D} where T^i 's form the basis of \mathcal{G} and t_i 's the corresponding dual basis of \mathcal{B} . We obtain

$$\begin{aligned} \{\phi, \psi\}_D &= \frac{1}{2} \langle \nabla^L \phi, T^i \rangle \langle \nabla^L \psi, t_i \rangle - \frac{1}{2} \langle \nabla^L \phi, t_i \rangle \langle \nabla^L \psi, T^i \rangle \\ &\quad + \frac{1}{2} \langle \nabla^R \phi, T^i \rangle \langle \nabla^R \psi, t_i \rangle - \frac{1}{2} \langle \nabla^R \phi, t_i \rangle \langle \nabla^R \psi, T^i \rangle, \end{aligned} \quad (4.16)$$

where the standard Einstein summation convention is used. By the duality of the basis t_i with respect to the basis T^i we mean that the following relation holds:

$$(t_i, T^j)_\mathcal{D} = \delta_i^j. \quad (4.17)$$

In order to prove the theorem 4.1, we need to handle in an efficient way the Semenov-Tian-Shansky symplectic form ω given by (4.8). We shall first prove the following lemma that will be used also for the proof of the theorem 4.6 of the next section.

Lemma 4.2: Consider a point $K \in D$ and four linear subspaces of the tangent space $T_K D$ defined as $S_L = L_{K*} \mathcal{G}$, $S_R = R_{K*} \mathcal{G}$, $\tilde{S}_L = L_{K*} \mathcal{B}$ and $\tilde{S}_R = R_{K*} \mathcal{B}$. Let $\Pi_{L\tilde{R}}$ be a projector on \tilde{S}_R with a kernel S_L and $\Pi_{\tilde{L}R}$ a projector on S_R with a kernel \tilde{S}_L . Then

$$\omega(t, u) = (t, (\Pi_{\tilde{L}R} - \Pi_{L\tilde{R}})u)_\mathcal{D}, \quad (4.18)$$

where t, u are arbitrary two vectors in the tangent space $T_K D$ at the point $K \in D$ and the metric $(\cdot, \cdot)_\mathcal{D}$ at the point K is defined by the right or left transport of the bilinear form $(\cdot, \cdot)_\mathcal{D}$ defined at the unit element $E \in D$.

Proof: First we rewrite (4.8) as

$$\omega = \frac{1}{2} (b_L^* \rho_B \frown \rho_D)_\mathcal{D} + \frac{1}{2} (b_R^* \lambda_B \frown \lambda_D)_\mathcal{D}, \quad (4.19)$$

where $\lambda_D(\rho_D)$ is the left (right) invariant Maurer-Cartan form on the double D . In order to see that (4.19) is correct, we write

$$\frac{1}{2} (dK K^{-1} \frown db_L b_L^{-1})_\mathcal{D} =$$

$$= \frac{1}{2}(db_L b_L^{-1} + b_L dg_R g_R^{-1} b_L^{-1} \frown db_L b_L^{-1})_{\mathcal{D}} = \frac{1}{2}(g_R^* \rho_G \frown b_L^* \lambda_B)_{\mathcal{D}} \quad (4.20)$$

and

$$\begin{aligned} & \frac{1}{2}(K^{-1} dK \frown b_R^{-1} db_R)_{\mathcal{D}} = \\ & = \frac{1}{2}(b_R^{-1} g_L^{-1} dg_L b_R + b_R^{-1} db_R \frown b_R^{-1} db_R)_{\mathcal{D}} = \frac{1}{2}(g_L^* \lambda_G \frown b_R^* \rho_B)_{\mathcal{D}}. \end{aligned} \quad (4.21)$$

Here we used $K = b_L(K)g_R(K)$ in the first relation (4.20) and $K = g_L(K)b_R(K)$ in the second one (4.21).

Take a vector $v \in S_L \subset T_K D$ and calculate the expression

$$\langle b_L^* \rho_B, v \rangle = R_{b_L^{-1}*} b_{L*} v = 0. \quad (4.22)$$

The vanishing of this expression follows from the fact that $b_L(K e^{sT}) = b_L(K)$ for every $T \in \mathcal{G}$. Now consider another vector $w \in \tilde{S}_R \subset T_K D$. We have

$$\langle b_L^* \rho_B, w \rangle = R_{b_L^{-1}*} b_{L*} w = R_{b_L^{-1}*} R_{g_R^{-1}*} w = R_{K^{-1}*} w. \quad (4.23)$$

This follows from the fact that $b_L(e^{st} K) = e^{st} b_L(K) = e^{st} K g_R^{-1}(K)$ for every $t \in \mathcal{B}$. We thus obtain for an arbitrary vector $u \in T_K D$ that

$$\langle b_L^* \rho_B, u \rangle = \langle b_L^* \rho_B, (\Pi_{\tilde{R}L} + \Pi_{L\tilde{R}})u \rangle = R_{b_L^{-1}*} b_{L*} \Pi_{L\tilde{R}} u = R_{K^{-1}*} \Pi_{L\tilde{R}} u. \quad (4.24)$$

Much in the same way as above we derive

$$\langle b_R^* \lambda_B, u \rangle = L_{K^{-1}*} \Pi_{R\tilde{L}} u. \quad (4.25)$$

Combining the formulas (4.19), (4.24) and (4.25), we arrive at

$$\begin{aligned} -2\omega(t, u) &= (R_{K^{-1}*} t, R_{K^{-1}*} \Pi_{L\tilde{R}} u)_{\mathcal{D}} + (L_{K^{-1}*} t, L_{K^{-1}*} \Pi_{R\tilde{L}} u)_{\mathcal{D}} \\ &\quad - (R_{K^{-1}*} u, R_{K^{-1}*} \Pi_{L\tilde{R}} t)_{\mathcal{D}} - (L_{K^{-1}*} u, L_{K^{-1}*} \Pi_{R\tilde{L}} t)_{\mathcal{D}} = \\ &= (t, \Pi_{L\tilde{R}} u)_{\mathcal{D}} + (t, \Pi_{R\tilde{L}} u)_{\mathcal{D}} - (u, \Pi_{L\tilde{R}} t)_{\mathcal{D}} - (u, \Pi_{R\tilde{L}} t)_{\mathcal{D}}. \end{aligned} \quad (4.26)$$

Now we use the obvious fact

$$\Pi_{L\tilde{R}} + \Pi_{\tilde{R}L} = 1, \quad \Pi_{R\tilde{L}} + \Pi_{\tilde{L}R} = 1$$

(here the first index of the projector denotes its kernel and the second its image) and obtain three relations

$$(t, \Pi_{R\tilde{L}}u)_{\mathcal{D}} = (t, u) - (t, \Pi_{\tilde{L}R}u)_{\mathcal{D}}; \quad (4.27)$$

$$-(u, \Pi_{L\tilde{R}}t)_{\mathcal{D}} = -(\Pi_{\tilde{R}L}u, \Pi_{L\tilde{R}}t)_{\mathcal{D}} = -(\Pi_{\tilde{R}L}u, t)_{\mathcal{D}} = -(u, t) + (\Pi_{L\tilde{R}}u, t)_{\mathcal{D}}; \quad (4.28)$$

$$-(u, \Pi_{R\tilde{L}}t)_{\mathcal{D}} = -(\Pi_{\tilde{L}R}u, \Pi_{R\tilde{L}}t)_{\mathcal{D}} = -(\Pi_{\tilde{L}R}u, t)_{\mathcal{D}}. \quad (4.29)$$

Inserting (4.27), (4.28) and (4.29) into (4.26), we obtain

$$\omega(t, u) = (t, (\Pi_{\tilde{L}R} - \Pi_{L\tilde{R}})u)_{\mathcal{D}}. \quad (4.30)$$

The lemma is proved.

#

Proof of the theorem 4.1: The strategy of the proof is as follows: Since the antisymmetry of the bracket (4.9) is obvious from (4.16), we have to prove only the validity of the Jacobi identity, in order to show that (4.9) is really the Poisson bracket. Then we shall prove the non-degeneracy of this Poisson structure by showing that the form ω is dual (inverse) to the Poisson bivector corresponding to the bracket (4.9).

The Poisson bracket (4.9) on D can be rewritten in terms of a certain bivector (antisymmetric two-tensor) α such that the following relation holds

$$\{\phi, \psi\}_D = \alpha(d\phi, d\psi), \quad \phi, \psi \in Fun(D). \quad (4.31)$$

We can easily identify α by noting that (4.16) can be rewritten as

$$\{\phi, \psi\}_D = \langle r, \nabla^L \phi \otimes \nabla^L \psi \rangle + \langle r, \nabla^R \phi \otimes \nabla^R \psi \rangle, \quad (4.32)$$

where $r \in \Lambda^2 \mathcal{D}$ is the so called classical r -matrix. It is clearly given by

$$r = \frac{1}{2}(T^i \otimes t_i - t_i \otimes T^i). \quad (4.33)$$

From this we have

$$\alpha_K = (L_{K*} + R_{K*})r, \quad K \in D. \quad (4.34)$$

The Jacobi identity for the Poisson bracket is equivalent to the following condition for the bivector α

$$[\alpha, \alpha]_S = 0. \quad (4.35)$$

Here $[\cdot, \cdot]_S$ is the Schouten bracket of the multivectors [1]. Recall its main properties

$$[\alpha, \beta]_S = -(-1)^{(|\alpha|-1)(|\beta|-1)}[\beta, \alpha]_S, \quad (4.36)$$

$$[\alpha, \beta \wedge \gamma]_S = [\alpha, \beta]_S \wedge \gamma + (-1)^{(|\alpha|-1)|\beta|}\beta \wedge [\alpha, \gamma]_S. \quad (4.37)$$

Moreover, for any vector field X , the bracket $[X, \alpha]_S$ is just the Lie derivative.

Let us calculate $[\alpha, \alpha]_S$ for α given by (4.34). Since right-invariant vector fields on the Lie group manifolds commute with left-invariant ones, the result reads

$$[\alpha, \alpha]_S = [L_{K*}r, L_{K*}r]_S + [R_{K*}r, R_{K*}r]_S = (L_{K*} - R_{K*})[r, r]_S. \quad (4.38)$$

Here we use the same symbol $[\cdot, \cdot]_S$ also for the Schouten bracket based on the Lie algebra commutator. Thus we see that the Jacobi identity is fulfilled iff $[r, r]_S$ is the invariant element of $\Lambda^3\mathcal{D}$. Actually, using (4.33), the direct calculation gives

$$[r, r]_S = ([t_i, t_l], T^k)_{\mathcal{D}} t_k \wedge T^i \wedge T^l + ([T^i, T^l], t_k)_{\mathcal{D}} T^k \wedge t_i \wedge t_l. \quad (4.39)$$

Using the bracket (4.5), the \mathcal{D} -invariance of this expression can be checked by the direct calculation. But another way to see it is to realize that the r.h.s. of (4.39) coincides with the invariant Cartan (or WZW) element of $\Lambda^3\mathcal{D}$ canonically associated to the invariant bilinear form $(\cdot, \cdot)_{\mathcal{D}}$ (cf. [1]).

It is well-known that the bivector α satisfying (4.35) defines a symplectic structure iff it exists a dual (or inverse) 2-form $\omega \in \Lambda^2 T^*D$. The latter is then automatically closed ($d\omega = 0$), as the consequence of (4.35). The duality means that

$$\alpha(\cdot, \omega(\cdot, u)) = u \quad (4.40)$$

for any vector field $u \in TD$. Let us prove (4.40) for α given by (4.34) and ω by (4.18).

First we note that the element $t_i \otimes T^i + T^i \otimes t_i \in \mathcal{D} \otimes \mathcal{D}$ is invariant since it corresponds to the invariant bilinear form $(\cdot, \cdot)_{\mathcal{D}}$. Due to this fact the expression for α can be rewritten as

$$\alpha = L_{K*}(T^i \otimes t_i) - R_{K*}(t_i \otimes T^i). \quad (4.41)$$

Thus we have

$$\alpha(., \omega(., u)) = R_{K*}t_i(R_{K*}T^i, (\Pi_{L\tilde{R}} - \Pi_{\tilde{L}R})u)_{\mathcal{D}} - L_{K*}T^i(L_{K*}t_i, (\Pi_{L\tilde{R}} - \Pi_{\tilde{L}R})u)_{\mathcal{D}}. \quad (4.42)$$

This can be rewritten as

$$\alpha(., \omega(., u)) = (\Pi_{R\tilde{R}} - \Pi_{\tilde{L}L})(\Pi_{L\tilde{R}} - \Pi_{\tilde{L}R})u. \quad (4.43)$$

Recall that the first subscript of the projector stands for the kernel and the second for the image (cf. Proof of Lemma 4.2). Now we have

$$\begin{aligned} & (\Pi_{R\tilde{R}} - \Pi_{\tilde{L}L})(\Pi_{L\tilde{R}} - \Pi_{\tilde{L}R}) = \Pi_{R\tilde{R}}\Pi_{L\tilde{R}} - \Pi_{\tilde{L}L}\Pi_{L\tilde{R}} + \Pi_{\tilde{L}L}\Pi_{\tilde{L}R} = \\ & = \Pi_{L\tilde{R}} - \Pi_{\tilde{L}L}\Pi_{L\tilde{R}} + \Pi_{\tilde{L}L} = (\Pi_{L\tilde{R}} + \Pi_{\tilde{R}L}) + (\Pi_{\tilde{L}L} - \Pi_{\tilde{R}L} - \Pi_{\tilde{L}L}\Pi_{L\tilde{R}}) = 1 + 0. \end{aligned} \quad (4.44)$$

Combining (4.43) and (4.44), we arrive finally at

$$\alpha(., \omega(., u)) = u. \quad (4.45)$$

The theorem is proved. #

Remark: In order to show that $d\omega = 0$, we can also use the Polyakov-Wiegmann formula [40] applied for $K = b_L(K)g_R(K)$ and $K = g_L(K)b_R(K)$ respectively:

$$\begin{aligned} & \frac{1}{3}(\rho_D \frown \rho_D \wedge \rho_D)_{\mathcal{D}} = d(g_R^*\rho_G \frown b_L^*\lambda_B)_{\mathcal{D}} + \\ & + \frac{1}{3}(b_L^*\rho_B \frown b_L^*\rho_B \wedge b_L^*\rho_B)_{\mathcal{D}} + \frac{1}{3}(g_R^*\rho_G \frown g_R^*\rho_G \wedge g_R^*\rho_G)_{\mathcal{D}}; \end{aligned} \quad (4.46)$$

$$\begin{aligned} & \frac{1}{3}(\rho_D \frown \rho_D \wedge \rho_D)_{\mathcal{D}} = d(b_R^*\rho_B \frown g_L^*\lambda_G)_{\mathcal{D}} + \\ & + \frac{1}{3}(b_R^*\rho_R \frown b_R^*\rho_B \wedge b_R^*\rho_B)_{\mathcal{D}} + \frac{1}{3}(g_L^*\rho_G \frown g_L^*\rho_G \wedge g_L^*\rho_G)_{\mathcal{D}}. \end{aligned} \quad (4.47)$$

Note that the last two terms in (4.46) and also in (4.47) vanish because of the isotropy of the Lie algebras \mathcal{G} and \mathcal{B} with respect to the bilinear form $(.,.)_{\mathcal{D}}$. Using (4.46), (4.47) and the definition (4.8) of the Semenov-Tian-Shansky form ω , we arrive at

$$-d\omega = \frac{1}{2}d(g_R^*\rho_G \frown b_L^*\lambda_B)_{\mathcal{D}} + \frac{1}{2}d(g_L^*\lambda_G \frown b_R^*\rho_B)_{\mathcal{D}} =$$

$$= \frac{1}{6}(\rho_D \hat{\lrcorner} \rho_D \wedge \rho_D)_{\mathcal{D}} - \frac{1}{6}(\rho_D \hat{\lrcorner} \rho_D \wedge \rho_D)_{\mathcal{D}} = 0. \quad (4.48)$$

We note also that physicists write the Polyakov-Wiegmann formula (4.46) as follows

$$\begin{aligned} & \frac{1}{3}(dKK^{-1} \hat{\lrcorner} dKK^{-1} \wedge dKK^{-1})_{\mathcal{D}} = d(dg_R g_R^{-1} \hat{\lrcorner} b_L^{-1} db_L)_{\mathcal{D}} + \\ & + \frac{1}{3}(db_L b_L^{-1} \hat{\lrcorner} db_L b_L^{-1} \wedge db_L b_L^{-1})_{\mathcal{D}} + \frac{1}{3}(dg_R g_R^{-1} \hat{\lrcorner} dg_R g_R^{-1} \wedge dg_R g_R^{-1})_{\mathcal{D}}. \end{aligned} \quad (4.49)$$

Definition 4.3: The **Heisenberg** double is the Drinfeld double equipped with the Poisson structure (4.9).

In what follows we shall always suppose that the multiplication maps $\mathcal{M}_{L,R}$ are bijective hence for us the Heisenberg double will always be the *symplectic* manifold.

The Poisson bracket on the Heisenberg double of G has the following crucial property:

Proposition 4.4: (Semenov-Tian-Shansky [44]): The algebra $Fun(D)^B$ of right B -invariant functions on the Heisenberg double D is a Lie subalgebra in the Poisson algebra (4.9) of all functions $Fun(D)$. The same is true for the algebra ${}^B Fun(D)$ of left B -invariant functions and also for the algebras $Fun(D)^G$ and ${}^G Fun(D)$.

Proof: Since the role of the groups G and B is completely symmetric in D , we shall restrict our attention only to the cases $Fun(D)^G$ and ${}^G Fun(D)$. Let $\rho, \eta \in Fun(D)^G$, which means that

$$\langle \nabla^R \rho, T^i \rangle = \langle \nabla^R \eta, T^i \rangle = 0. \quad (4.50)$$

From (4.16) we see immediately that

$$\{\rho, \eta\}_D = \frac{1}{2} \langle \nabla^L \rho, T^i \rangle \langle \nabla^L \eta, t_i \rangle - \frac{1}{2} \langle \nabla^L \rho, t_i \rangle \langle \nabla^L \eta, T^i \rangle = \frac{1}{2} (\nabla^L \rho, \mathcal{R}^* \nabla^L \eta)_{\mathcal{D}^*}. \quad (4.51)$$

Now having in mind that the left and right derivatives ∇^L and ∇^R commute with each other, we obtain

$$\langle \nabla^R \{\rho, \eta\}_D, T^i \rangle = 0. \quad (4.52)$$

The case ${}^G Fun(D)$ can be treated in the same way. The proposition is proved.

This proposition permits to construct the Poisson-Lie brackets on B (and on G). Indeed, both $Fun(D)^G$ and ${}^G Fun(D)$ are naturally isomorphic to $Fun(B)$ (due to the existence of the global decomposition $D = GB = BG$) and the following proposition holds

Proposition 4.5: The Poisson brackets on $Fun(B)$ induced from the Semenov-Tian-Shansky bracket on $Fun(D)^G$ and ${}^G Fun(D)$ coincide up to sign and they verify the Poisson-Lie condition (4.2).

Proof: Take two functions Φ and Ψ in $Fun(B)$ and calculate their Poisson bracket $\{\Phi, \Psi\}_B^R$ induced from $Fun(D)^G$. We have by definition

$$\begin{aligned} b_L^* \{\Phi, \Psi\}_B^R &= \{b_L^* \Phi, b_L^* \Psi\}_D = \\ &= \frac{1}{2} \langle \nabla_D^L(b_L^* \Phi), T^i \rangle \langle \nabla_D^L(b_L^* \Psi), t_i \rangle - \frac{1}{2} \langle \nabla_D^L(b_L^* \Phi), t_i \rangle \langle \nabla_D^L(b_L^* \Psi), T^i \rangle. \end{aligned} \quad (4.53)$$

The superscript R over the Poisson bracket on B indicates that the latter originates from $Fun(D)^G$. Now it is obvious that

$$\langle \nabla_D^L(b_L^* \Psi), t_i \rangle = b_L^* \langle \nabla_B^L \Psi, t_i \rangle, \quad (4.54)$$

where the subscripts D and B indicates the group on which the differential operators live. It is less straightforward to evaluate $\langle \nabla_D^L(b_L^* \Phi), T^i \rangle$.

We shall proceed as follows: we note that a map $b_L \circ \mathcal{M}_L : G \times B \rightarrow B$ given by

$$g \times b \rightarrow b_L(gb) \equiv Dres_g b \quad (4.55)$$

defines a left action of the group G on the manifold B . It is easy to see this since the following relation holds

$$b_L(g_1 g_2 b) = b_L(g_1 b_L(g_2 b) g_R(g_2 b)) = b_L(g_1 b_L(g_2 b)), \quad g_1, g_2 \in G, \quad b \in B. \quad (4.56)$$

Now by definition

$$\langle \nabla_D^L(b_L^* \Phi), T^i \rangle(K) = \left(\frac{d}{ds} \right)_{s=0} \Phi(b_L(e^{sT^i} b_L(K))), \quad K \in D. \quad (4.57)$$

Looking at the relations (4.56) and (4.57), we see that there exists a vector field (a differential operator) ∇_B^i acting on $Fun(B)$ such that

$$\langle \nabla_D^L(b_L^* \Phi), T^i \rangle = b_L^*(\nabla_B^i \Phi). \quad (4.58)$$

Such operator ∇_B^i can be certainly expressed as a linear combination of the operators $\langle \nabla_B^L, t_j \rangle$. In other words, there exists a matrix valued function $\Pi_R^{ij}(b)$ on B such that

$$\nabla_B^i \Phi = \Pi_R^{ij}(b) \langle \nabla_B^L \Phi, t_j \rangle. \quad (4.59)$$

Hence from (4.53), we obtain for our (right) Poisson bracket the following expression

$$\{\Phi, \Psi\}_B^R = \frac{1}{2} \langle \nabla_B^L \Phi, t_i \rangle (\Pi_R^{ij}(b) - \Pi_R^{ji}(b)) \langle \nabla_B^L \Psi, t_j \rangle. \quad (4.60)$$

Proceeding in the same way, we define the left bracket

$$b_R^* \{\Phi, \Psi\}_B^L = \{b_R^* \Phi, b_R^* \Psi\}_D \quad (4.61)$$

and we have

$$\{\Phi, \Psi\}_B^L = \frac{1}{2} \langle \nabla_B^R \Phi, t_i \rangle (\Pi_L^{ij}(b) - \Pi_L^{ji}(b)) \langle \nabla_B^R \Psi, t_j \rangle \quad (4.62)$$

for certain matrix valued function $\Pi_L^{ij}(b)$ on B . In order to show that the left Poisson bracket (4.60) is equal to the minus right one (4.62), we have to know more about the matrices $\Pi_{L,R}^{ij}$. Introduce first the following matrices (cf. [35])

$$A_i^j(K) = (K^{-1} t_i K, T^j)_{\mathcal{D}}, \quad B^{ij}(K) = (K^{-1} T^i K, T^j)_{\mathcal{D}}, \quad K \in D. \quad (4.63)$$

Now calculate

$$T^i b = b b^{-1} T^i b = b(B^{ij}(b) t_j + A_j^i(b^{-1}) T^j). \quad (4.64)$$

Since the differential operator $\langle \nabla_B^R, t_j \rangle$ corresponds to the vector field $L_{b^*} t_j = b t_j$, we see from (4.64) that the operator ∇_B^i can be expressed as

$$\nabla_B^i = B^{ij}(b) \langle \nabla_B^R, t_j \rangle = B^{ik}(b) A_k^j(b^{-1}) \langle \nabla_B^L, t_j \rangle. \quad (4.65)$$

Hence

$$\Pi_R^{ij}(b) = -B^{ik}(b)A_k^j(b^{-1}). \quad (4.66)$$

On the other hand, we have

$$bT^i = bT^ib^{-1}b = (B^{ij}(b^{-1})t_j + A_j^i(b)T^j)b \quad (4.67)$$

and from this we arrive at

$$\Pi_L^{ij}(b) = -B^{ik}(b^{-1})A_k^j(b) = -B^{ki}(b)A_k^j(b). \quad (4.68)$$

Let us now show that the tensors Π_L^{ij} and Π_R^{ij} are antisymmetric. We have

$$\begin{aligned} -\Pi_L^{ij}(b) &= B^{ki}(b)A_k^j(b) = (b^{-1}T^kb, T^i)_{\mathcal{D}}(b^{-1}t_kb, T^j)_{\mathcal{D}} = \\ &= (Ad_bT^i, T^k)_{\mathcal{D}}(t_k, Ad_bT^j) = -(Ad_bT^i, t_k)_{\mathcal{D}}(T^k, Ad_bT^j) = \Pi_L^{ji}(b) \end{aligned} \quad (4.69)$$

and similarly for Π_R^{ij} . Thus the expressions (4.60) and (4.62) simplify as follows

$$\{\Phi, \Psi\}_B^R = \langle \nabla_B^L \Phi, t_i \rangle \Pi_R^{ij}(b) \langle \nabla_B^L \Psi, t_j \rangle \quad (4.70)$$

$$\{\Phi, \Psi\}_B^L = \langle \nabla_B^R \Phi, t_i \rangle \Pi_L^{ij}(b) \langle \nabla_B^R \Psi, t_j \rangle \quad (4.71)$$

Using the obvious relation

$$\langle \nabla_B^L, t_i \rangle = A_i^j(b) \langle \nabla_B^R, t_j \rangle \quad (4.72)$$

we can rewrite the right bracket as

$$\{\Phi, \Psi\}_B^R = \langle \nabla_B^R \Phi, t_m \rangle A_i^m(b) \Pi_R^{ij}(b) A_j^l(b) \langle \nabla_B^R \Psi, t_l \rangle. \quad (4.73)$$

Now we have

$$A_i^m(b) \Pi_R^{ij}(b) A_j^l(b) = -A_i^m(b) B^{ik}(b) A_k^j(b^{-1}) A_j^l(b) = -A_i^m(b) B^{il}(b) = \Pi_L^{lm}(b). \quad (4.74)$$

From (4.70), (4.71) and (4.74), we conclude that

$$\{\Phi, \Psi\}_B^R = -\{\Phi, \Psi\}_B^L. \quad (4.75)$$

It remains to show that the Poisson bracket $\{.,.\}_B^R$ verifies the Poisson-Lie condition (4.2). Recall that a Poisson bracket on any manifold can be equivalently described by a Poisson bivector (=antisymmetric tensor) $\alpha \in \Lambda^2 TB$; the Poisson-Lie bracket on B in terms of α is given by

$$\{\phi, \psi\}_B = \langle \alpha, d\phi \otimes d\psi \rangle, \quad \phi, \psi \in Fun(B). \quad (4.76)$$

The Poisson-Lie condition (4.2) can be directly rewritten as

$$\alpha_{ab} = L_{a*}\alpha_b + R_{b*}\alpha_a, \quad a, b \in B, \quad (4.77)$$

where $\alpha_b \in \Lambda^2 T_b B$. Since the bivector bundle $\Lambda^2 T B$ on any group manifold is trivializable by the right-invariant vector fields, we loose no information about the Poisson-Lie structure α if we trade it for another object, namely a map $\Pi : G \rightarrow \Lambda^2 \mathcal{G}$ defined as follows

$$\Pi_R(b) = R_{b^{-1}*}\alpha_b. \quad (4.78)$$

Now $\Pi_R(b)$ can be expressed in the basis t_i as

$$\Pi_R(b) = \Pi_R^{ij}(b) t_i \otimes t_j. \quad (4.79)$$

In fact, the matrix Π_R^{ij} just introduced is the same as the one in (4.59) (the notation is thus consistent!) because the operator $\langle \nabla^L, t_i \rangle$ corresponds to the vector field defined in each point $b \in B$ as $R_{b*}t_i$. The condition (4.77) translates in terms of $\Pi_R(b)$ as

$$\Pi_R(ab) = \Pi_R(a) + Ad_a \Pi_R(b). \quad (4.80)$$

One can directly check that the expression (4.66) fulfils (4.80), by using the definition (4.63) of the matrices $A_i^j(b)$ and $B^{ij}(b)$.

The theorem is proved.

#

4.1.3 Lu-Weinstein-Soibelman double

We are now going to present two important examples of the Heisenberg doubles. We shall present a third important (loop group) example in section 4.4.

1) The cotangent bundle T^*G of any Lie group is its Heisenberg double. The bilinear form $(\cdot, \cdot)_{\mathcal{D}}$ is defined as in (7.22). The role of the group B is played by the subgroup of all elements K of T^*G for which $P_K = e$. Looking at the T^*G multiplication law (7.18), we immediately discover that B is an Abelian group, in fact, it is nothing but the dual vector space space \mathcal{G}^* of the Lie algebra \mathcal{G} of G . The global decomposability $D = GB = BG$ follows from the fact that the cotangent bundle of any Lie group is trivializable. The

Poisson-Lie bracket on G , induced from the bracket on the Heisenberg double T^*G , identically vanishes. This follows from (7.44). On the other hand, the Poisson-Lie bracket on B is nontrivial. From (7.53), we see that it is in fact the Kirillov-Kostant Poisson bracket on \mathcal{G}^* .

2) Consider now any finite-dimensional simple compact connected Lie group G_0 . For its Heisenberg double D we take simply its complexification (viewed as the *real* group) $G_0^{\mathbf{C}}$ of G_0 . So, for example, the double of $SU(2)$ is $SL(2, \mathbf{C})$. The invariant non-degenerate form $(\cdot, \cdot)_{\mathcal{D}}$ on the Lie algebra $\mathcal{D} = \mathcal{G}_0^{\mathbf{C}}$ of $D = G_0^{\mathbf{C}}$ is given by

$$(x, y)_{\mathcal{D}} = \frac{1}{\varepsilon} \text{Im}(x, y)_{\mathcal{G}_0^{\mathbf{C}}}, \quad (4.81)$$

or, in other words, it is just the imaginary part of the Killing-Cartan form $(\cdot, \cdot)_{\mathcal{G}_0^{\mathbf{C}}}$ divided by a real parameter ε . We shall see later that ε plays the role of a deformation parameter. Since G_0 is the compact real form of $G_0^{\mathbf{C}}$, clearly the imaginary part of $(x, y)_{\mathcal{G}_0^{\mathbf{C}}}$ vanishes if $x, y \in \mathcal{G}_0$. Hence, G_0 is indeed isotropically embedded in $G_0^{\mathbf{C}}$. The double $G_0^{\mathbf{C}}$ equipped with the metric (4.81) is usually referred to as the double of Lu & Weinstein [38] and of Soibelman [45].

It turns out that $G_0^{\mathbf{C}}$ is indeed the Drinfeld double, because $D = G_0^{\mathbf{C}}$ is at the same time the double of its another subgroup B_0 which coincides with the so called AN group in the Iwasawa decomposition of $G_0^{\mathbf{C}}$:

$$G_0^{\mathbf{C}} = G_0 AN = AN G_0. \quad (4.82)$$

For the groups $SL(n, \mathbf{C})$, the group AN can be identified with the upper triangular matrices of determinant 1 and with positive real numbers on the diagonal. In general, the elements of AN can be uniquely represented by means of the exponential map as follows

$$\tilde{g} = e^{\phi} \exp[\sum_{\alpha > 0} v_{\alpha} E^{\alpha}] \equiv e^{\phi} n. \quad (4.83)$$

Here α 's denote the roots of $\mathcal{G}_0^{\mathbf{C}}$, v_{α} are complex numbers, E^{α} are the step operators and ϕ is an Hermitian element¹ of the Cartan subalgebra of $\mathcal{G}_0^{\mathbf{C}}$.

¹Recall that the Hermitian element of any complex simple Lie algebra $\mathcal{G}^{\mathbf{C}}$ is an eigenvector of the involution which defines the compact real form \mathcal{G} ; the corresponding eigenvalue is (-1) . This involution originates from the group involution $g \rightarrow (g^{-1})^{\dagger}$. The anti-Hermitian elements that span the compact real form are eigenvectors of the same involution with the eigenvalue equal to 1. For elements of $sl(n, \mathbf{C})$ Lie algebra, the Hermitian element is indeed a Hermitian matrix in the standard sense.

Loosely said, A is the "noncompact part" of the complex maximal torus of $G_0^{\mathbb{C}}$. The isotropy of the Lie algebra \mathcal{B}_0 of $B_0 = AN$ follows from (4.81); the fact that \mathcal{G}_0 and \mathcal{B}_0 generate together the Lie algebra \mathcal{D} of the whole double is evident from (4.82). The Iwasawa decomposition itself is the global decomposition $D = G_0 B_0 = B_0 G_0$ needed for ensuring that the Semenov-Tian-Shansky Poisson bracket (4.9) does indeed define the (everywhere non-degenerate) symplectic structure on D .

4.1.4 Non-Abelian moment maps

The concept of a Poisson-Lie symmetry [44] is the generalization of the traditional Hamiltonian symmetry of a dynamical system defined by a symplectic manifold and a Hamiltonian. Here we shall partially follow the exposition of the papers [20, 4] and [36].

First we need to recall the definition of the **dressing action** of G on its dual Poisson-Lie group B . An element $g \in G$ acts on an element $b \in B$ to give

$$Dres_g b \equiv b_L(gb), \quad (4.84)$$

where the multiplication gb is taken in the Drinfeld double D and the map $b_L : D \rightarrow B$ is induced by the decomposition $D = BG$. It follows from (4.56) that this is really the group action, i.e.

$$Dres_e b = b, \quad Dres_{(g_1 g_2)} b = Dres_{g_1}(Dres_{g_2} b). \quad (4.85)$$

Suppose now that there is a symplectic form ω on a manifold P and there is a left action of a Poisson-Lie group G on P , infinitesimally generated by a section v of the bundle $TP \otimes \mathcal{G}^* = TP \otimes \mathcal{B}$. The vector field corresponding to the action of a generator $T \in \mathcal{G}$ is then $\langle v, T \rangle \in TP$. Recall that B is the dual Poisson-Lie group and \mathcal{B} is its Lie algebra; the dual space \mathcal{G}^* is identified with \mathcal{B} via the invariant bilinear form $(\cdot, \cdot)_{\mathcal{D}}$ on the Drinfeld double $\mathcal{D} = \mathcal{G} + \mathcal{B}$ (cf. section 4.1.1 and 4.1.2).

Moreover, suppose that there is a G -equivariant map $M : P \rightarrow B$, where G acts on P as above and acts on B via the dressing action. Finally, such a map M is called the non-Abelian moment map if it holds

$$-i_v \omega \equiv \omega(\cdot, v) = M^* \rho_B. \quad (4.86)$$

Said in words: the contraction of the symplectic form ω by the section $v \in TP \otimes \mathcal{B}$ is equal to the pull-back of the right-invariant Maurer-Cartan form ρ_B on B by the map M . The condition (4.86) is often written as

$$\omega(., v) = dMM^{-1}. \quad (4.87)$$

Theorem 4.6: Let the manifold P be the Heisenberg double D of G and ω is the Semenov-Tian-Shansky symplectic form (4.8). Then

- 1) The map $b_L : D \rightarrow B$ induced by the decomposition $D = BG$ is the non-Abelian moment map M of the standard left action $G \times D \rightarrow D$ given by the group multiplication law in D , i.e. $(g, K) \rightarrow gK$, $g \in G, K \in D$.
- 2) The map $b_R^{-1} : D \rightarrow B$ induced by the decomposition $D = GB$ is the non-Abelian moment map M of the left action $G \times D \rightarrow D$ given by $(g, K) \rightarrow Kg^{-1}$, $g \in G, K \in D$.

Proof: The first part 1) is slightly easier: We have to show, that for a generator $T \in \mathcal{G}$, it holds

$$\omega(u, R_{K*}T) = (T, \langle b_L^* \rho_B, u \rangle)_{\mathcal{D}}, \quad (4.88)$$

where $K \in D$ and $u \in T_K D$. We have

$$\omega(u, R_{K*}T) = (R_{K*}T, (\Pi_{L\tilde{R}} - \Pi_{\tilde{L}R})u)_{\mathcal{D}} = (R_{K*}T, \Pi_{L\tilde{R}}u)_{\mathcal{D}} = (T, \langle b_L^* \rho_B, u \rangle)_{\mathcal{D}}. \quad (4.89)$$

Here the first equality follows from the Lemma 4.2, the second from the isotropy of the space S_R with respect to $(.,.)_{\mathcal{D}}$ and the third one from the relation (4.24).

For the second part 2), consider again the generator $T \in \mathcal{G}$. We want to show, that

$$\omega(L_{K*}T, u) = (T, \langle d(b_R^{-1})(b_R^{-1})^{-1}, u \rangle)_{\mathcal{D}}. \quad (4.90)$$

The last relation can be rewritten as

$$\omega(u, L_{K*}T) = (T, \langle b_R^{-1}db_R, u \rangle)_{\mathcal{D}} = (T, \langle b_R^* \lambda_B, u \rangle)_{\mathcal{D}}. \quad (4.91)$$

Let us prove (4.91). We have

$$\omega(u, L_{K*}T) = (L_{K*}T, (\Pi_{L\tilde{R}} - \Pi_{\tilde{L}R})u)_{\mathcal{D}} =$$

$$\begin{aligned}
&= (L_{K*}T, (\Pi_{L\tilde{R}} + \Pi_{\tilde{R}L})u)_{\mathcal{D}} - (L_{K*}T, (\Pi_{\tilde{L}R} + \Pi_{R\tilde{L}})u)_{\mathcal{D}} + (L_{K*}T, \Pi_{R\tilde{L}}u)_{\mathcal{D}} = \\
&= (L_{K*}T, u)_{\mathcal{D}} - (L_{K*}T, u)_{\mathcal{D}} + (L_{K*}T, \Pi_{R\tilde{L}}u)_{\mathcal{D}} = \\
&= (L_{K*}T, \Pi_{R\tilde{L}}u)_{\mathcal{D}} = (T, \langle b_R^* \lambda_B, u \rangle)_{\mathcal{D}}. \tag{4.92}
\end{aligned}$$

The last equality in (4.92) follows from (4.). The G -equivariance of M is obvious in the first case 1); in the second it follows from the relation

$$b_R^{-1}(K) = b_L(K^{-1}), \quad K \in D. \tag{4.93}$$

The theorem is proved.

#

We end up this section with a definition:

Definition 4.7: A dynamical system characterized by a symplectic manifold P , a symplectic form ω and a Hamiltonian H is Poisson-Lie symmetric with respect to a left action of a Poisson-Lie group G , if the Hamiltonian is G -invariant and the G -action on P is generated by the non-Abelian moment map $M : P \rightarrow B$ fulfilling the condition (4.86).

Remark: The classical action of any dynamical system can be written in the standard way

$$S = \int (\theta - H dt), \tag{4.94}$$

where $d\theta$ is the symplectic form and H the Hamiltonian. The variation of the action of a general Poisson-Lie symmetric system was calculated in [4] with the result

$$\delta_T S = - \int \epsilon_i A^i. \tag{4.95}$$

Here A^i is the set of functions on the phase space P , satisfying the non-Abelian zero-curvature condition

$$dA^i = \tilde{f}^i_{kl} A^k A^l; \tag{4.96}$$

ϵ_i are the coefficients of the generator $T \in \mathcal{G}$ in some basis T^i of \mathcal{G} and \tilde{f}^i_{kl} are the structure constants of \mathcal{B} in the dual basis t_i .

If $A^i = dX^i$ for some collection of functions X^i on P , then \tilde{f}^i_{kl} obviously vanishes and the integrand on the r.h.s. of (4.95) is the total derivative. The action S is then strictly symmetric with respect to the G -action and X^i are nothing but the standard Hamiltonian charges generating the symmetry. However, if \tilde{f}^i_{kl} do not vanish, the action S is not symmetric in the strict sense of this word. However, it is by definition Poisson-Lie symmetric since its variation has the special form encoded in (4.95) and (4.96).

4.2 Quasitriangular geodesical model

Definition 4.8: Consider now a Lie group G and let D be some of its Heisenberg doubles. Choose an appropriate G -biinvariant Hamiltonian function $H(K)$, $K \in D$. The dynamical system defined by the Semenov-Tian-Shansky symplectic structure on D and by the biinvariant Hamiltonian $H(K)$ will be called the quasitriangular geodesical model.

We know from the results of the previous section, that the quasitriangular geodesical model is distinguished by two independent Poisson-Lie symmetries given by the left multiplication $k_L K$, $k_L \in G$ or the right multiplication (but the left action!) $K k_R^{-1}$, $k_R \in G$. Even without knowing anything more about the Hamiltonian $H(K)$, its G -biinvariance entails immediately an important information about the trajectories of this dynamical system (we shall call such a trajectory a quasitriangular geodesics). Indeed: Let $K(t)$ be the quasitriangular geodesics. Then

$$b_L(K(t)) = b_L(K(0)) = b_L^0, \quad b_R(K(t)) = b_R(K(0)) = b_R^0. \quad (4.97)$$

In other words, b_L^0 and b_R^0 are nonlinear constant of motions. We may say loosely that the nonlinear momentum b_L (or b_R) is constant, and the quasitriangular geodesical motion is therefore "nonlinearly free" in the nonlinear coordinate g_R (or g_L).

The independence of b_L and b_R on time is the consequence of the G -biinvariance of $H(K)$ and of the following theorem:

Theorem 4.9: The Semenov-Tian-Shansky Poisson bracket of a G -leftinvariant function on D with a G -rightinvariant function on D always vanishes.

Proof: Due to the existence of the global decompositions $D = GB = BG$, each G -rightinvariant function on D is a b_L -pullback of some function $\Phi \in Fun(B)$ and similarly each G -leftinvariant function on D is a b_R -pullback of some function $\Psi \in Fun(B)$. Using the formula (4.41), the Semenov-Tian-Shansky bracket (4.9) of such two functions then becomes

$$\{b_L^* \Phi, b_R^* \Psi\}_D = \frac{1}{2} \langle \nabla_D^R(b_L^* \Phi), T^i \rangle \langle \nabla_D^R(b_R^* \Psi), t_i \rangle - \frac{1}{2} \langle \nabla_D^L(b_L^* \Phi), t_i \rangle \langle \nabla_D^L(b_R^* \Psi), T^i \rangle. \quad (4.98)$$

Now the left (right) G -invariance of $b_R^* \Psi$ ($b_L^* \Phi$) means, respectively,

$$\langle \nabla_D^L(b_R^* \Psi), T^i \rangle(K) = 0, \quad (4.99)$$

$$\langle \nabla_D^R(b_L^* \Phi), T^i \rangle(K) = 0. \quad (4.100)$$

Thus

$$\{b_L^* \Phi, b_R^* \Psi\}_D = 0. \quad (4.101)$$

The theorem is proved. #

As an example, we can take for G the simple compact connected group G_0 and its Heisenberg double is the Lu-Weinstein-Soibelman double $D_0 = G_0^{\mathbf{C}}$ of the example 2) of the previous paragraph. Now we look for a biinvariant Hamiltonian. Recall that in the case of the standard geodesical model the choice of the biinvariant Hamiltonian was canonical. It turns out that in the $D_0 = G_0^{\mathbf{C}}$ case, there is also the canonical choice. Up to a normalization (to be fixed later), it is given by

$$H = (\ln(b_L^\dagger b_L), \ln(b_L^\dagger b_L))_{\mathcal{G}_0^{\mathbf{C}}} = (\ln(b_R^\dagger b_R), \ln(b_R^\dagger b_R))_{\mathcal{G}_0^{\mathbf{C}}}, \quad (4.102)$$

where \dagger is the Hermitian conjugation defined in the section 4.1.3. We shall return to the quasitriangular geodesical model on $G_0^{\mathbf{C}}$ later when we shall study its chiral decomposition. It will be on this occasion when we shall give the exact solution of the model.

4.3 WZW Drinfeld doubles of \tilde{G}

In this section, \tilde{G} will denote the central biextension of the group G . Recall that in the non-deformed case, we have performed the two-step symplectic reduction starting from the geodesical model (1.1) on \tilde{G} with its canonical symplectic structure on $T^*\tilde{G}$ and arriving at the WZW model on G with its (non-canonical) WZW symplectic form on T^*G . In principle, we can construct the deformation of the master model (1.1) for whatever Drinfeld double of \tilde{G} . However, if we want to make also the two-step symplectic reduction, the double must be of the special form described in the following definition:

Definition 4.10: We shall say that a double $\tilde{D}(= \tilde{G}\tilde{B} = \tilde{B}\tilde{G})$ of the central biextension \tilde{G} is of the WZW type (or simply is the WZW double), iff there exist a double $\hat{D} = (\hat{G}\hat{B} = \hat{B}\hat{G})$ of \hat{G} and a double $D(= GB = BG)$ of G such that

- i) The group \tilde{B} is isomorphic to the direct product of the real line with \hat{B} , i.e. $\tilde{B} = \mathbf{R} \times \hat{B}$.
- ii) The group \hat{B} is isomorphic to the semi-direct product of the real line with B , i.e. $\hat{B} = \mathbf{R} \times_Q B$, where Q is some one parameter group of automorphisms of B .

Theorem 4.11: The properties 1)-2) of the definition 4.10 are satisfied by the triple of Drinfeld doubles $\tilde{D} = T^*\tilde{G}$, $\hat{D} = T^*\hat{G}$ and $D = T^*G$.

Proof: The dual group \tilde{B} is clearly \tilde{G}^* viewed as the Abelian group (linear space). Their elements are $(A, \alpha, a)^*$ (cf. Conventions 2.5). The subgroups formed by the elements of the forms $(0, \alpha, a)^*$ and $(0, \alpha, 0)^*$ are \hat{B} and B , respectively. Since \tilde{B} is Abelian, the theorem is proved.

#

In order to give the definition of the universal quasitriangular WZW model, it is useful to set appropriate conventions:

Conventions 4.12: Consider the group $\tilde{B} = \mathbf{R} \times \hat{B} = \mathbf{R} \times (\mathbf{R} \times_Q B)$ from the definition 4.10. The symbol \times without (with) the subscript means the direct (semidirect) product of groups. This decomposition induces several natural maps; our conventions will give them names. First of all, from $\tilde{B} = \mathbf{R} \times \hat{B}$ we obtain two natural maps $m^0 : \tilde{B} \rightarrow \mathbf{R}$ and $\hat{m} : \tilde{B} \rightarrow \hat{B}$. Then from $\hat{B} = \mathbf{R} \times_Q B$ we produce $m^\infty : \hat{B} \rightarrow \mathbf{R}$ and two maps $m_{L,R} : \hat{B} \rightarrow B$ given by two possible decompositions $\hat{B} = B\mathbf{R} = \mathbf{R}B$. Recall also that the notation $\tilde{b}_{L,R}$ is induced by the two canonical decompositions $\tilde{D} = \tilde{G}\tilde{B} = \tilde{B}\tilde{G}$.

Definition 4.13: The universal quasitriangular WZW model is the rule that associates a dynamical system to every WZW Drinfeld double \tilde{D} (of the central biextension \tilde{G}) and to every \tilde{G} -biinvariant function $\tilde{H}(\tilde{K})$, $\tilde{K} \in \tilde{D}$. This system is obtained from the quasitriangular geodesical model on \tilde{D} by

the two-step symplectic reduction induced by setting

$$m^0(\tilde{b}_L(\tilde{K})) + m^0(\tilde{b}_R(\tilde{K})) = 0; \quad (4.103)$$

$$(m^\infty \circ \hat{m})(\tilde{b}_L(\tilde{K})) + (m^\infty \circ \hat{m})(\tilde{b}_R(\tilde{K})) = 2\kappa. \quad (4.104)$$

We shall call this dynamical system the (\tilde{D}, \tilde{H}) -quasitriangular WZW model.

Remarks:

1) For a general WZW double \tilde{D} , we do not have a natural choice of Hamiltonian \tilde{H} . However, two important WZW doubles of the affine Kac-Moody group \tilde{G} permit to choose the Hamiltonian in the canonical way. The first one is the cotangent bundle $T^*\tilde{G}$ which leads to the definition of the standard loop group WZW model described in Chapters 2 and 3. The second one is the affine Lu-Weinstein-Soibelman double which will be introduced in the next Section and which will lead to the main result of this paper: the construction of the loop group quasitriangular WZW model.

2) It is important to note that the moment maps appearing in the relations (4.103) and (4.104) are ordinary functions. They generate respectively the axial actions (i.e. $\tilde{K} \rightarrow u\tilde{K}u$) of the group \mathbf{R}_S generated by \tilde{T}^0 and of the central circle S^1 generated by \tilde{T}^∞ . For a non-WZW double of \tilde{G} such axial action would be only of the Poisson-Lie type and, generically, we would not be able to disentangle the moment maps of the \tilde{T}^0 and \tilde{T}^∞ symmetries from the moment maps of the other \tilde{G} -symmetries. The choice of the WZW double insures that these particular symmetries are Hamiltonian in the standard sense of this word hence the symplectic reduction can be performed.

3) We shall describe the details of the symplectic reduction for the affine Lu-Weinstein-Soibelman double of the affine Kac-Moody group. There is no interest to list here explicitly the reduction for whatever WZW double since the corresponding formulas are anyway too general to be illuminating.

4.4 Affine Lu-Weinstein-Soibelman double

Now we are advancing to our most important example of the triple (\tilde{D}, \hat{D}, D) of the Drinfeld doubles of the groups (\tilde{G}, \hat{G}, G) (cf. Definition 4.10). G will be the loop group $G = LG_0$, where G_0 is a simple compact connected and simply connected Lie group²; $\hat{G} = \widehat{LG}_0$ will be its standard central

²Note that G is then connected.

extension described in Appendix 7.1 and \tilde{G} will be the affine Kac-Moody group $\mathbf{R} \times_{\tilde{s}} \widehat{LG}_0$. Although the construction that we are going to present here is apparently original, we choose the name "affine Lu-Weinstein-Soibelman double" because the resulting double \tilde{D} has many features similar as the finite-dimensional Lu-Weinstein-Soibelman double $D_0 = G_0^{\mathbf{C}}$.

4.4.1 The double $D = LG_0^{\mathbf{C}}$

For the Heisenberg double D of G , we take the loop group $LG_0^{\mathbf{C}}$ consisting of smooth maps from the circle S^1 into $G_0^{\mathbf{C}}$. It will be often convenient to view the loop group $LG_0^{\mathbf{C}}$ as a group of holomorphic maps from the σ -Riemann sphere without poles into the complex group $G_0^{\mathbf{C}}$. Clearly, the loop circle S^1 is identified with the equator. The σ -Riemann sphere is in fact the ordinary Riemann sphere but since in this section we shall encounter the notion of the Riemann sphere in several different contexts, we shall use the labels to distinguish them.

The complex Lie algebra $\mathcal{G}^{\mathbf{C}} = \widehat{L\mathcal{G}_0^{\mathbf{C}}}$ is equipped with an invariant non-degenerate bilinear form given by

$$(x, y)_{\mathcal{G}^{\mathbf{C}}} = \frac{1}{2\pi} \int d\sigma (x(\sigma), y(\sigma))_{\mathcal{G}_0^{\mathbf{C}}}, \quad (4.105)$$

where the elements $x, y \in \mathcal{G}^{\mathbf{C}}$ are smooth maps from S^1 into $\mathcal{G}_0^{\mathbf{C}}$. Recall that D is the group $\widehat{LG_0^{\mathbf{C}}}$ viewed as the real group. The invariant nondegenerate bilinear form on $\mathcal{D} = Lie(D)$ is then defined as

$$(x, y)_{\mathcal{D}} = \frac{1}{\varepsilon} Im(x, y)_{\mathcal{G}^{\mathbf{C}}}, \quad (4.106)$$

where ε is a real positive parameter. Note the full analogy with the finite-dimensional definition (4.81). The Lie algebra $\mathcal{G} = L\mathcal{G}_0$ is isotropic with respect to $(\cdot, \cdot)_{\mathcal{D}}$ since it is "pointwise" isotropic with respect to $Im(\cdot, \cdot)_{\mathcal{G}_0^{\mathbf{C}}}$. Thus we see that D is indeed the Manin double of G . In order to show that it is also the Drinfeld double, we need the complementary (or dual) group B . In fact, B is the group $L_+G_0^{\mathbf{C}}$ consisting of loops in $G_0^{\mathbf{C}}$ that are boundary values of holomorphic maps from the unit disc into $G_0^{\mathbf{C}}$. In other words, we may view $L_+G_0^{\mathbf{C}}$ as the group of holomorphic maps from the σ -Riemann sphere without the north pole into the complex group $G_0^{\mathbf{C}}$. We require moreover,

that the value of this holomorphic map at the origin of the disc (=the south pole of the σ -Riemann sphere) is an element of $B_0 = AN \in G_0^{\mathbf{C}}$.

As we have already said, the isotropy of the Lie algebra LG_0 of $G = LG_0$ with respect to the bilinear form (4.106) is obvious. But also the Lie algebra $\mathcal{B} = L_+\mathcal{G}_0^{\mathbf{C}}$ is isotropic with respect to $(\cdot, \cdot)_{\mathcal{D}}$. Indeed, the expression to be integrated in (4.106) has only the non-negative Fourier modes. The integral of all strictly positive modes then vanishes. The zero mode does not contribute either since

$$Im(Lie(AN), Lie(AN))_{\mathcal{G}_0} = 0 \quad (4.107)$$

as in the example 2) of Section 4.13. Finally, the existence of the global decomposition $D = LG_0^{\mathbf{C}} = (LG_0)(L_+G_0^{\mathbf{C}}) = GB = BG$ was proved in [41].

4.4.2 The double $\hat{D} = \mathbf{R} \times_Q \mathbf{R}\widehat{LG_0^{\mathbf{C}}}$

Now we are going to construct the double \hat{D} of the centrally extended loop group \hat{G} .

Consider the group $DG_0^{\mathbf{C}}$ of smooth maps from the unit disc into $G_0^{\mathbf{C}}$ with the usual pointwise multiplication. We can now define an extended group $\mathbf{R}\widehat{DG_0^{\mathbf{C}}}$ whose elements are pairs (\bar{l}, λ) where $\bar{l} \in DG_0^{\mathbf{C}}$ and $\lambda \in U(1)$ and whose multiplication law reads

$$(\bar{l}_1, \lambda_1)(\bar{l}_2, \lambda_2) = (\bar{l}_1\bar{l}_2, \lambda_1\lambda_2 \exp[2\pi i\beta_{\mathbf{R}}(\bar{l}_1, \bar{l}_2)]). \quad (4.108)$$

Here $\beta_{\mathbf{R}}$ is a real valued 2-cocycle on $DG_0^{\mathbf{C}}$ given by

$$\beta_{\mathbf{R}}(\bar{l}_1, \bar{l}_2) = \frac{1}{8\pi^2} \int_{Disc} Re(\bar{l}_1^{-1} d\bar{l}_1 \wedge d\bar{l}_2 \bar{l}_2^{-1})_{\mathcal{G}_0^{\mathbf{C}}}, \quad (4.109)$$

where $(\cdot, \cdot)_{\mathcal{G}_0^{\mathbf{C}}}$ is again the (standardly normalized) Killing-Cartan form on $\mathcal{G}_0^{\mathbf{C}}$. It is crucial to note the presence of the real part symbol in the definition of the 2-cocycle. This real part is also reflected by the (left) superscript \mathbf{R} in the symbol $\mathbf{R}\widehat{DG_0^{\mathbf{C}}}$.

Consider now a subgroup $\partial G^{\mathbf{C}}$ of $DG_0^{\mathbf{C}}$ consisting of all smooth maps from the *Disc* into $G_0^{\mathbf{C}}$ such that their value at every point of the boundary $\partial D = S^1$ is the unit element e_0 of $G_0^{\mathbf{C}}$. Any $\bar{l} \in \partial G^{\mathbf{C}}$ can be thought of as a map $\bar{l}: S^2 \rightarrow G_0^{\mathbf{C}}$ by identifying the boundary S^1 of *Disc* with the north pole of S^2 . The Riemann sphere thus obtained will be called the *D*-Riemann

sphere. It turns out that there is a homomorphism $\Theta_{\mathbf{R}}^{\mathbf{C}} : \partial G^{\mathbf{C}} \rightarrow {}^{\mathbf{R}}\widehat{DG}_0^{\mathbf{C}}$ defined by

$$\Theta_{\mathbf{R}}^{\mathbf{C}}(\bar{l}) = (\bar{l}, \exp[-2\pi i C_{\mathbf{R}}^{\mathbf{C}}(\bar{l})]), \quad (4.110)$$

where

$$C_{\mathbf{R}}^{\mathbf{C}}(\bar{l}) = \frac{1}{24\pi^2} \int_{Ball} Re(d\bar{l}l^{-1} \frown d\bar{l}l^{-1} \wedge d\bar{l}l^{-1})|_{\mathcal{G}_0^{\mathbf{C}}}. \quad (4.111)$$

Here *Ball* is the unit ball whose boundary is the *D*-Riemann sphere and we have extended the map $\bar{l} : S^2 \rightarrow G_0^{\mathbf{C}}$ to a map $\bar{l} : Ball \rightarrow G_0^{\mathbf{C}}$. The proof of the fact that $\exp[-2\pi i C_{\mathbf{R}}^{\mathbf{C}}(\bar{l})]$ does not depend on the extension of \bar{l} to *Ball* reduces to the same proof as for the compact group G_0 since $G_0^{\mathbf{C}}$ has the same homotopies as G_0 . The demonstration that $\Theta_{\mathbf{R}}^{\mathbf{C}}$ is indeed a homomorphism and the fact that the image $\Theta_{\mathbf{R}}^{\mathbf{C}}(\partial G^{\mathbf{C}})$ is the normal subgroup in ${}^{\mathbf{R}}\widehat{DG}_0^{\mathbf{C}}$ follows again from the Polyakov-Wiegmann formula [40] (see Appendix 7.1) which asserts that

$$C_{\mathbf{R}}^{\mathbf{C}}(\bar{l}_1 \bar{l}_2) = C_{\mathbf{R}}^{\mathbf{C}}(\bar{l}_1) + C_{\mathbf{R}}^{\mathbf{C}}(\bar{l}_2) - \beta_{\mathbf{R}}(\bar{l}_1, \bar{l}_2). \quad (4.112)$$

The subgroup $\hat{D} \equiv {}^{\mathbf{R}}\widehat{LG}_0^{\mathbf{C}}$ of the double $\hat{\hat{D}}$ is now defined as the factor group ${}^{\mathbf{R}}\widehat{DG}_0^{\mathbf{C}}/\Theta_{\mathbf{R}}^{\mathbf{C}}(\partial G^{\mathbf{C}})$. This group is a (nontrivial) circle bundle over the base space $LG_0^{\mathbf{C}} = D$. The projection Π_0 is $(\bar{l}, \lambda) \rightarrow \bar{l}|_{S^1}$ and the center of \hat{D} is represented by the Θ -equivalence classes represented by $(1, \lambda) \in {}^{\mathbf{R}}\widehat{DG}_0^{\mathbf{C}}$. The projection homomorphism from ${}^{\mathbf{R}}\widehat{DG}_0^{\mathbf{C}}$ onto \hat{D} will be referred to as $\wp_{\mathbf{C}}$.

In order to construct the Drinfeld double $\hat{\hat{D}}$, we need a one-parameter group $\mathbf{R}_{\hat{Q}}$ of automorphisms of \hat{D} . For this, it is convenient first to define a one-parameter group of automorphisms $\mathbf{R}_{\bar{Q}}$ of ${}^{\mathbf{R}}\widehat{DG}_0^{\mathbf{C}}$. If $(\bar{l}(z, r), \lambda)$ is an element in ${}^{\mathbf{R}}\widehat{DG}_0^{\mathbf{C}}$ the action of an element $w \in \mathbf{R}_{\bar{Q}}$ reads

$${}^q(\bar{l}(z, r), \lambda) = (\bar{l}(qz, r), \lambda), \quad (4.113)$$

where $q = e^w$. Recall, that we view the loop group $LG_0^{\mathbf{C}}$ as the group of holomorphic maps from the Riemann sphere without poles into the complex group $G_0^{\mathbf{C}}$. The standard polar coordinates (σ, r) of the *Disc* thus get traded for (z, r) . We can view the disc as the intersection of the equatorial plane with the interior of the σ -Riemann sphere. We stress however that this σ -Riemann sphere is not the same as the *D*-sphere used for the definition of the term $\exp[-2\pi i C_{\mathbf{R}}^{\mathbf{C}}(\bar{l})]$ for elements $\bar{l} \in \partial G^{\mathbf{C}}$.

We have to prove that

$${}^q((\bar{l}_1, \lambda_1)(\bar{l}_2, \lambda_2)) = {}^q(\bar{l}_1, \lambda_1){}^q(\bar{l}_2, \lambda_2), \quad (4.114)$$

where the product is considered in ${}^{\mathbf{R}}\widehat{DG}_0^{\mathbf{C}}$. This in turn amounts to show that

$$\beta_{\mathbf{R}}(\bar{l}_1(z, r), \bar{l}_2(z, r)) = \beta_{\mathbf{R}}(\bar{l}_1(qz, r), \bar{l}_2(qz, r)). \quad (4.115)$$

Recalling the definition of the cocycle $\beta_{\mathbf{R}}$, we can rewrite (4.109) as

$$\begin{aligned} \beta_{\mathbf{R}}(\bar{l}_1, \bar{l}_2) &= \\ &= \frac{1}{8\pi^2} \int_0^1 dr \int_{|z|=1} dz (Re(\bar{l}_1^{-1} \partial_r \bar{l}_1, \partial_z \bar{l}_2 \bar{l}_2^{-1})_{\mathcal{G}_0^{\mathbf{C}}} - Re(\bar{l}_1^{-1} \partial_z \bar{l}_1, \partial_r \bar{l}_2 \bar{l}_2^{-1})_{\mathcal{G}_0^{\mathbf{C}}}). \end{aligned} \quad (4.116)$$

Here the integration over σ is replaced by the contour integration along the equator $|z| = 1$ on the σ -Riemann sphere. It is straightforward to show that $\beta_{\mathbf{R}}(\bar{l}_1(qz, r), \bar{l}_2(qz, r))$ is given by the same integral as (4.116) but along the new contour $|z| = q$. Since the integrand is everywhere holomorphic function between the two contours, they can be deformed one to the other and we conclude that (4.115) holds.

Now we have to show that the group action (4.113) survives the factorization by the group $\delta G^{\mathbf{C}}$. In other words, the automorphisms \bar{Q} of ${}^{\mathbf{R}}\widehat{DG}_0^{\mathbf{C}}$ descends to automorphisms \hat{Q} of the factor group $\hat{D} = {}^{\mathbf{R}}\widehat{LG}_0^{\mathbf{C}}$, or still rephrased differently: \bar{Q} acts on the $\partial G^{\mathbf{C}}$ classes in ${}^{\mathbf{R}}\widehat{DG}_0^{\mathbf{C}}$. This amounts to show that

$$C_{\mathbf{R}}^{\mathbf{C}}(\bar{l}(z, r)) = C_{\mathbf{R}}^{\mathbf{C}}(\bar{l}(qz, r)). \quad (4.117)$$

Looking at the formula (4.111), the integration goes now over three variables parametrizing the interior of the D -Riemann sphere. We can again change the σ -integration to the contour integration on the σ -Riemann sphere and then we prove (4.117) by the identical contour deformation argument as above.

Thus our group $\hat{\hat{D}}$ is defined as the group of couples (X, q) , $X \in {}^{\mathbf{R}}\widehat{LG}_0^{\mathbf{C}}$, $q \in \mathbf{R}^+$ with the following composition law

$$(X_1, q_1)(X_2, q_2) = (X_1 {}^{q_1}X_2, q_1 q_2). \quad (4.118)$$

Now we have to prove several lemmas in order to show that $\hat{\hat{D}}$ is indeed the Drinfeld double of \hat{G} .

Lemma 4.14: The subgroup $\Pi_0^{-1}(LG_0) \subset \hat{D}$ is isomorphic to $\widehat{LG_0} = \hat{G}$.

Proof: Consider an element $\hat{l} \in \Pi_0^{-1}(LG_0) \subset {}^{\mathbf{R}}\widehat{LG_0^{\mathbf{C}}}$. It can be lifted by the "map" $\wp_{\mathbf{C}}^{-1}$ (of course non-uniquely) to some element $\bar{l} \in \widehat{DG_0} \subset {}^{\mathbf{R}}\widehat{DG_0^{\mathbf{C}}}$. We stress that the element \bar{l} can be chosen in $\widehat{DG_0}$. The element \bar{l} can be then projected by the map \wp (not $\wp_{\mathbf{C}}$!) to some element of $\widehat{LG_0}$. We now show that this element of $\widehat{LG_0}$ does not depend on the choice $\bar{l} \in \widehat{DG_0}$, which means that we have constructed certain map $\mu : \Pi_0^{-1}(LG_0) \rightarrow \widehat{LG_0}$. Indeed, if we have two elements $\bar{l}_1, \bar{l}_2 \in \widehat{DG_0}$ such that $\wp_{\mathbf{C}}(\bar{l}_1) = \wp_{\mathbf{C}}(\bar{l}_2) = \hat{l}$ then it certainly exists an element $F \in \partial G (\subset \partial G^{\mathbf{C}})$ such that $\bar{l}_1 = F\bar{l}_2$. This means, in other words, that $\wp(\bar{l}_1) = \wp(\bar{l}_2)$ hence μ is a well-defined map.

Let us show that μ is a homomorphism. First of all the unit element in Π_0^{-1} can be lifted directly to the unit element of $\widehat{DG_0}$ and mapped by \wp to the unit element of $\widehat{LG_0}$ (since \wp is a homomorphism). Then we want to prove that

$$\mu(\hat{l}_1 \hat{l}_2) = \mu(\hat{l}_1) \mu(\hat{l}_2), \quad \hat{l}_1, \hat{l}_2 \in \Pi_0^{-1}(LG_0).$$

But if $\bar{l}_1, \bar{l}_2 \in \widehat{DG_0}$ are such that $\wp_{\mathbf{C}}(\bar{l}_i) = \hat{l}_i; i = 1, 2$ then $\wp_{\mathbf{C}}(\bar{l}_1 \bar{l}_2) = \hat{l}_1 \hat{l}_2$ since $\wp_{\mathbf{C}}$ is the homomorphism. Thus we have

$$\mu(\hat{l}_1 \hat{l}_2) = \wp(\bar{l}_1 \bar{l}_2) = \wp(\bar{l}_1) \wp(\bar{l}_2) = \mu(\hat{l}_1) \mu(\hat{l}_2)$$

because also \wp is the homomorphism.

Injectivity: if we take again $\hat{l}_1, \hat{l}_2 \in \Pi_0^{-1}(LG_0)$ such that $\Pi_0(\hat{l}_1) \neq \Pi_0(\hat{l}_2)$ then clearly $\pi(\mu(\hat{l}_1)) \neq \pi(\mu(\hat{l}_2))$ hence $\mu(\hat{l}_1) \neq \mu(\hat{l}_2)$ (recall that π denotes the homomorphism from $\widehat{LG_0}$ to LG_0 defined by the exact sequence (2.1)). If $\Pi_0(\hat{l}_1) = \Pi_0(\hat{l}_2)$ and $\hat{l}_1 \neq \hat{l}_2$ then $\hat{l}_1 = Y\hat{l}_2$, where Y is a non-unit central circle element in ${}^{\mathbf{R}}\widehat{LG_0^{\mathbf{C}}}$. Then also $\bar{l}_{1,2} \in \widehat{DG_0}$ can be chosen to be connected by the same non-unit central circle element viewed as the element of $\widehat{DG_0}$ and $\wp(\bar{l}_1) = \mu(\hat{l}_1) = Y\mu(\hat{l}_2)$, where now the same Y is viewed as the element of $\widehat{LG_0}$.

Since the surjectivity is evident, the lemma is proved.

#

Lemma 4.15: The group $B = L_+ G_0^{\mathbf{C}}$ can be homomorphically injected into $\hat{D} = {}^{\mathbf{R}}\widehat{LG_0^{\mathbf{C}}}$. Moreover, the image of this injection is preserved by the automorphisms \hat{Q} .

Proof: Consider an element $b \in L_+G_0^{\mathbf{C}}$. By definition, it is the boundary value of the holomorphic map \bar{b} from the unit disc into $G_0^{\mathbf{C}}$. Consider now the map $\bar{\nu} : L_+G_0^{\mathbf{C}} \rightarrow {}^{\mathbf{R}}\widehat{DG}_0^{\mathbf{C}}$ defined by

$$\bar{\nu}(b) = (\bar{b}, 1). \quad (4.119)$$

The crucial thing is that the map $\bar{\nu}$ is the homomorphism of groups. This follows from the fact that the cocycle $\beta_{\mathbf{R}}(\bar{b}_1, \bar{b}_2)$ vanishes if \bar{b}_i 's are the holomorphic maps. Indeed, by using the contour representation (4.116) of this cocycle, we see immediately that the contour can be contracted to the origin of the unit disc without encountering any singularity because the integrated function is everywhere holomorphic.

Consider now the map $\nu : L_+G_0^{\mathbf{C}} \rightarrow \hat{D} = {}^{\mathbf{R}}\widehat{LG}_0^{\mathbf{C}}$ defined by

$$\nu = \wp_{\mathbf{C}} \circ \bar{\nu}. \quad (4.120)$$

First of all, ν is the group homomorphism being the composition of two homomorphisms. Moreover, it holds

$$(\Pi_0 \circ \wp_{\mathbf{C}} \circ \bar{\nu})(b) = b \in LG_0^{\mathbf{C}}. \quad (4.121)$$

From this it follows that ν is the injection. The invariance of $\nu(B)$ under the action of \hat{Q} is obvious.

The lemma is proved.

#

Remark: The two preceding lemmas say, in other words, that both $\hat{G} = \widehat{LG}_0$ and $\hat{B} = \mathbf{R} \times_Q L_+G_0^{\mathbf{C}}$ are subgroups of \hat{D} which is one of the basic properties of the Heisenberg double of \hat{G} and of \hat{B} .

Lemma 4.16: The group \hat{D} can be globally decomposed as $\hat{D} = \hat{G}\hat{B} = \hat{B}\hat{G}$ where $\hat{G} = \widehat{LG}_0$ and $\hat{B} = \mathbf{R} \times_Q L_+G_0^{\mathbf{C}}$.

Proof: Of course, in the sense of the two lemmas above, here $\Pi_0^{-1}(LG_0) \subset \hat{D}$ is viewed as \hat{G} and $\mathbf{R} \times_Q \nu(L_+G_0^{\mathbf{C}})$ as \hat{B} . Take any element \hat{K} in the extended double \hat{D} . Since $\hat{D} = \mathbf{R} \times_Q \hat{D}$, \hat{K} can be uniquely decomposed as

$$\hat{K} = (\hat{K}, 1)(1, q), \quad (4.122)$$

where $\hat{K} \in \hat{D}$ and $q = e^w$, $w \in \mathbf{R}_Q$.

Now we have to prove that \hat{K} can be uniquely decomposed as

$$\hat{K} = \hat{a}\nu(b), \quad \hat{a} \in \hat{G} = \Pi_0^{-1}(LG_0), \quad b \in B = L_+G_0^{\mathbf{C}}. \quad (4.123)$$

Consider first the element $\Pi_0(\hat{K}) \in LG_0^{\mathbf{C}}$. It can be uniquely decomposed as

$$\Pi_0(\hat{K}) = ab, \quad a \in LG_0, \quad b \in L_+G_0^{\mathbf{C}}. \quad (4.124)$$

Now it is certainly true that \hat{K} can be found among the elements of the circle fiber $\Pi_0^{-1}(a)\nu(b)$ above ab . The existence of \hat{a} and b from (4.123) then follows immediately.

It remains to be proved that the decomposition (4.123) is unique. So if

$$\hat{K} = \hat{a}\nu(b) = \hat{a}'\nu(b'), \quad a, a' \in \hat{G}, \quad b, b' \in B, \quad (4.125)$$

then

$$\Pi_0(\hat{a}\nu(b)) = \Pi_0(\hat{a})\Pi_0(\nu(b)) = \Pi_0(\hat{a})b = \Pi_0(\hat{a}')b'. \quad (4.126)$$

The second equality here follows from (4.121). It is now clear that $b = b'$ because they are both in $L_+G_0^{\mathbf{C}}$ and the decomposition of the element $\Pi_0(\hat{a}\nu(b)) \in LG_0^{\mathbf{C}}$ into elements of LG_0 and $L_+G_0^{\mathbf{C}}$ is unique. We conclude that $b = b'$, hence $\hat{a} = \hat{a}'$.

The lemma is proved.

#

Lemma 4.17: The group $\hat{D} = \mathbf{R} \times_Q \widehat{\mathbf{R}LG_0^{\mathbf{C}}}$ is indeed the common Heisenberg double of the groups LG_0 and $\mathbf{R} \times_Q L_+G_0^{\mathbf{C}}$, if we define the invariant bilinear form on its Lie algebra $\hat{\mathcal{D}}$ as

$$(\hat{i}(x), \hat{i}(y))_{\hat{\mathcal{D}}} = \frac{1}{2\pi\varepsilon} \int d\sigma \text{Im}(x(\sigma), y(\sigma))_{\mathcal{G}_0^{\mathbf{C}}} = (x, y)_{\mathcal{D}}, \quad (4.127)$$

$$(\hat{T}^\infty, \hat{i}(\mathcal{D}))_{\hat{\mathcal{D}}} = (\hat{t}_\infty^1, \hat{i}(\mathcal{D}))_{\hat{\mathcal{D}}} = 0; \quad (\hat{T}^\infty, \hat{t}_\infty^1)_{\hat{\mathcal{D}}} = \frac{1}{\varepsilon}, \quad (4.128)$$

Here \hat{t}_∞^1 is the generator of \mathbf{R}_Q ; \hat{T}^∞ that of the central $U(1)$ and $\hat{i} : \mathcal{D} \rightarrow \hat{\mathcal{D}}$ is the natural extension of the injection $\iota : \mathcal{G} \rightarrow \hat{\mathcal{G}}$. The injection \hat{i} acting on \mathcal{B} is given by the derivation map ν_* .

Proof: We know already that $\hat{\hat{D}}$ contains both $\hat{G} = L\hat{G}_0$ and $\hat{B} = \mathbf{R} \times_Q L_+G_0^{\mathbf{C}}$ as its subgroups and that the global decomposition $\hat{D} = \hat{G}\hat{B} = \hat{B}\hat{G}$ takes place. The isotropy of the corresponding Lie subalgebras $\hat{\mathcal{G}}$ and $\hat{\mathcal{B}}$ follows from the isotropy of \mathcal{G} and \mathcal{B} with respect to the form $(\cdot, \cdot)_{\mathcal{D}}$. Indeed, for instance, $\hat{\mathcal{G}} = \iota(\mathcal{G}) + \text{Span}(\hat{T}^\infty)$ then for $\xi, \eta \in \mathcal{G}$ we have

$$(\iota(\xi), \iota(\eta))_{\hat{\hat{D}}} = (\xi, \eta)_{\mathcal{D}} = 0.$$

Adding to this the fact (4.128) that $(\hat{T}^\infty, \iota(\mathcal{G}))_{\hat{\hat{D}}} = 0$ we obtain $(\hat{\mathcal{G}}, \hat{\mathcal{G}})_{\hat{\hat{D}}} = 0$.

The bilinear form $(\cdot, \cdot)_{\hat{\hat{D}}}$ is also symmetric (if we complete appropriately the part (4.128) of its definition) since the form $(\cdot, \cdot)_{\mathcal{D}}$ is symmetric. The non-degeneracy of the form $(\cdot, \cdot)_{\hat{\hat{D}}}$ also follows immediately from the defining relations (4.127), (4.128) and from the fact that $(\cdot, \cdot)_{\mathcal{D}}$ is non-degenerate.

The remaining thing to show is the invariance of the form $(\cdot, \cdot)_{\hat{\hat{D}}}$. This can be derived by the direct calculation from the Lie bracket in $\hat{\hat{D}}$:

$$\begin{aligned} & [w_1\hat{T}^\infty + \xi_1^c(\sigma) + \varrho_1\hat{t}_\infty^1, w_2\hat{T}^\infty + \xi_2^c(\sigma) + \varrho_2\hat{t}_\infty^1] = \\ & = +\frac{1}{2\pi} \int d\sigma \text{Re}(\xi_1^c, \partial_\sigma \xi_2^c) \hat{T}^\infty + [\xi_1^c, \xi_2^c](\sigma) - \varrho_1 i \partial_\sigma \xi_2^c(\sigma) + \varrho_2 i \partial_\sigma \xi_1^c(\sigma). \end{aligned}$$

Here w_i, ϱ_i are real numbers and $\xi_i^c(\sigma)$ are from $L\mathcal{G}_0^{\mathbf{C}}$.

The lemma is proved.

#

Remark: We observe that $\hat{B} = \mathbf{R} \times_Q B$ as required by the Definition 4.10.

4.4.3 The double $\tilde{\hat{D}} = \mathbf{R}^2 \times_{S,Q} \widehat{LG}_0^{\mathbf{C}}$

As the title of this subsection suggests, the double $\tilde{\hat{D}}$ of the affine Kac-Moody group \tilde{G} will be the semidirect product of the plane \mathbf{R}^2 with certain group $\widehat{LG}_0^{\mathbf{C}}$. In order to construct $\widehat{LG}_0^{\mathbf{C}}$, we shall proceed in close analogy with the previous section and our intermediate explanations will be therefore much briefer.

Consider the group $DG_0^{\mathbf{C}}$ of smooth maps from the unit disc into $G_0^{\mathbf{C}}$ with the usual pointwise multiplication. We can now define an extended group

$\widehat{DG}_0^{\mathbf{C}}$ whose elements are pairs (\bar{l}, v) where $\bar{l} \in DG_0^{\mathbf{C}}$ and $v \in \mathbf{C}^\times$ and whose multiplication law reads

$$(\bar{l}_1, v_1)(\bar{l}_2, v_2) = (\bar{l}_1 \bar{l}_2, v_1 v_2 \exp [2\pi i \beta(\bar{l}_1, \bar{l}_2)]). \quad (4.129)$$

Here \mathbf{C}^\times is the complex plane without the origin viewed as the Abelian multiplicative group of complex numbers and β is a complex-valued 2-cocycle on $DG_0^{\mathbf{C}}$ given by

$$\beta(\bar{l}_1, \bar{l}_2) = \frac{1}{8\pi^2} \int_{Disc} (\bar{l}_1^{-1} d\bar{l}_1 \wedge d\bar{l}_2 \bar{l}_2^{-1})_{\mathcal{G}_0^{\mathbf{C}}}. \quad (4.130)$$

The remaining notations are the same as in the previous section. It is crucial to note the absence of the real part symbol in the definition of the 2-cocycle.

Consider now the subgroup $\partial G^{\mathbf{C}}$ of $DG_0^{\mathbf{C}}$. It turns out that there is a homomorphism $\Theta^{\mathbf{C}} : \partial G^{\mathbf{C}} \rightarrow \widehat{DG}_0^{\mathbf{C}}$ defined by

$$\Theta^{\mathbf{C}}(\bar{l}) = (\bar{l}, \exp [-2\pi i C^{\mathbf{C}}(\bar{l})]), \quad (4.131)$$

where

$$C^{\mathbf{C}}(\bar{l}) = \frac{1}{24\pi^2} \int_{Ball} (d\bar{l} \bar{l}^{-1} \wedge d\bar{l} \bar{l}^{-1} \wedge d\bar{l} \bar{l}^{-1})_{\mathcal{G}_0^{\mathbf{C}}}. \quad (4.132)$$

The group $\widehat{LG}_0^{\mathbf{C}}$ is now defined as the factor group $\widehat{DG}_0^{\mathbf{C}} / \Theta^{\mathbf{C}}(\partial G^{\mathbf{C}})$. This group is a (nontrivial) \mathbf{C}^\times bundle over the base space $LG_0^{\mathbf{C}} = D$. The projection $\Pi_0^{\mathbf{C}}$ is $(\bar{l}, v) \rightarrow \bar{l}|_{S^1}$ and the center of $\widehat{LG}_0^{\mathbf{C}}$ is represented by the $\Theta^{\mathbf{C}}$ -equivalence classes represented by $(1, v) \in \widehat{DG}_0^{\mathbf{C}}$. The projection homomorphism from $\widehat{DG}_0^{\mathbf{C}}$ onto $\widehat{LG}_0^{\mathbf{C}}$ will be referred to as $\hat{\rho}_{\mathbf{C}}$.

In order to construct the Drinfeld double $\tilde{\tilde{D}}$, we need two commuting one-parameter groups of automorphisms of $\widehat{LG}_0^{\mathbf{C}}$. We shall denote them as $\mathbf{R}_{\hat{Q}}$ and $\mathbf{R}_{\hat{S}}$. To define them, associate first to every complex number $w + is$ an automorphism of $\widehat{DG}_0^{\mathbf{C}}$ given by

$${}^q(\bar{l}(z, r), v) = (\bar{l}(qz, r), v), \quad (4.133)$$

where $q = \exp(w + is)$ and $\bar{l}(z, r) \in DG_0^{\mathbf{C}}$. Recall, that we view the loop group $LG_0^{\mathbf{C}}$ as the group of holomorphic maps from the σ -Riemann sphere without poles into the complex group $G_0^{\mathbf{C}}$. Similarly as in Section 4.4.2, we can prove that

$${}^q((\bar{l}_1, v_1)(\bar{l}_2, v_2)) = {}^q(\bar{l}_1, v_1){}^q(\bar{l}_2, v_2), \quad (4.134)$$

and that the group action (4.133) survives the factorization by the group $\partial G^{\mathbf{C}}$.

Thus our group \tilde{D} is defined as the group of couples (X, q) , $X \in \widehat{LG}_0^{\mathbf{C}}$, $q \in \mathbf{C}^\times$ with the following composition law

$$(X_1, q_1)(X_2, q_2) = (X_1 \cdot {}^{q_1}X_2, q_1 q_2). \quad (4.135)$$

Now we have to prove several lemmas in order to show that \tilde{D} is indeed the Drinfeld double of \tilde{G} .

Lemma 4.18: The group $\tilde{G} = \mathbf{R} \times_S \hat{G}$ is the subgroup of \tilde{D} .

Proof: First we prove that $\hat{G} = \widehat{LG}_0$ is the subset of $\widehat{LG}_0^{\mathbf{C}}$. Consider the set

$$\mathcal{S} = \{\hat{l} \in \widehat{LG}_0^{\mathbf{C}}, \exists \bar{l} \in \widehat{DG}_0 \subset \widehat{DG}_0^{\mathbf{C}}, \quad \wp_{\mathbf{C}}(\bar{l}) = \hat{l}\}. \quad (4.136)$$

We are going to show that the set \mathcal{S} is the subgroup of $\widehat{LG}_0^{\mathbf{C}}$ isomorphic to \widehat{LG}_0 . The isomorphism $\mu : \mathcal{S} \rightarrow \widehat{LG}_0$ is defined as follows:

$$\mu(\hat{l}) = \wp(\bar{l}). \quad (4.137)$$

Recall that $\wp : \widehat{DG}_0 \rightarrow \widehat{LG}_0$ is the map associating to every element of \widehat{DG}_0 its Θ -class (cf. Section 7.1). It is immediate to check that the definition of μ does not depend on the choice of the representative $\bar{l} \in \widehat{DG}_0$.

If \bar{l}_1, \bar{l}_2 are the respective representatives of \hat{l}_1, \hat{l}_2 , then obviously $\bar{l}_1 \bar{l}_2$ can be chosen as the representative of $\hat{l}_1 \hat{l}_2$. Since $\bar{l}_1 \bar{l}_2 \in \widehat{DG}_0$, it follows that the set \mathcal{S} is the subgroup of $\widehat{LG}_0^{\mathbf{C}}$. Moreover one has

$$\wp(\bar{l}_1 \bar{l}_2) = \wp(\bar{l}_1) \wp(\bar{l}_2), \quad (4.138)$$

hence μ is the group homomorphism. It remains to show the injectivity and surjectivity of μ .

If $\hat{l}_1 \neq \hat{l}_2$ and $\Pi_0^{\mathbf{C}}(\hat{l}_1) \neq \Pi_0^{\mathbf{C}}(\hat{l}_2)$ then obviously $\wp(\bar{l}_1) \neq \wp(\bar{l}_2)$. If $\hat{l}_1 \neq \hat{l}_2$ and $\Pi_0^{\mathbf{C}}(\hat{l}_1) = \Pi_0^{\mathbf{C}}(\hat{l}_2)$ then we can choose \bar{l}_1, \bar{l}_2 in such a way that

$$\bar{l}_1 = \bar{l}_2(1, e^{i\phi}), \quad e^{i\phi} \neq 1. \quad (4.139)$$

Then obviously $\wp(\bar{l}_1) \neq \wp(\bar{l}_2)$ and the injectivity follows. The surjectivity is also clear since the central circle acts freely on \mathcal{S} .

The lemma is proved. #

Lemma 4.19: The group $B = L_+G_0^{\mathbf{C}}$ can be homomorphically injected into $\widehat{LG_0^{\mathbf{C}}}$. Moreover, the image of this injection is preserved by the automorphisms \hat{Q} .

Proof: Consider an element $b \in L_+G_0^{\mathbf{C}}$. By definition, it is the boundary value of the holomorphic map \bar{b} from the unit disc into $G_0^{\mathbf{C}}$. Consider now the map $\bar{\nu} : L_+G_0^{\mathbf{C}} \rightarrow \widehat{DG_0^{\mathbf{C}}}$ defined by

$$\bar{\nu}(b) = (\bar{b}, 1). \quad (4.140)$$

As in the proof of Lemma 4.15, it can be shown that the map $\bar{\nu}$ is the homomorphism of groups.

Consider now the map $\hat{\nu} : L_+G_0^{\mathbf{C}} \rightarrow \widehat{LG_0^{\mathbf{C}}}$ defined by

$$\hat{\nu} = \hat{\wp}_{\mathbf{C}} \circ \bar{\nu}. \quad (4.141)$$

First of all, $\hat{\nu}$ is the group homomorphism being the composition of two homomorphisms. Moreover, it holds

$$(\Pi_0^{\mathbf{C}} \circ \wp_{\mathbf{C}} \circ \bar{\nu})(b) = b \in LG_0^{\mathbf{C}}. \quad (4.142)$$

From this it follows that $\hat{\nu}$ is the injection. The invariance of $\hat{\nu}(B)$ under the action of \hat{Q} is obvious.

The lemma is proved. #

Remark: The two preceding lemmas say, in other words, that both $\tilde{G} = \mathbf{R} \times_S \hat{G}$ and $\tilde{B} = (\mathbf{R} \times_Q L_+G_0^{\mathbf{C}}) \times \mathbf{R}$ are subgroups of $\tilde{\tilde{D}}$ which is one of the basic properties of the Heisenberg double of \tilde{G} and of \tilde{B} . Of course, the direct product factor \mathbf{R} here is the central line subgroup of $\widehat{LG_0^{\mathbf{C}}}$ corresponding to $\Theta^{\mathbf{C}}$ -classes in $\widehat{DG_0^{\mathbf{C}}}$ of the form $(1, e^t), t \in \mathbf{R}$.

Lemma 4.20: The group $\tilde{\tilde{D}}$ can be globally decomposed as $\tilde{\tilde{D}} = \tilde{G}\tilde{B} = \tilde{B}\tilde{G}$ where $\tilde{G} = \mathbf{R} \times_S \widehat{LG_0}$ and $\tilde{B} = (\mathbf{R} \times_Q L_+G_0^{\mathbf{C}}) \times \mathbf{R}$.

Proof: Of course, it is sufficient to prove that $\widehat{LG_0^C}$ can be decomposed as $\widehat{LG_0^C} = \hat{G}(\hat{\nu}(B) \times \mathbf{R})$, where \mathbf{R} stands for the central line in the sense of the remark above. Thus we are going to prove that $\hat{K} \in \widehat{LG_0^C}$ can be uniquely decomposed as

$$\hat{K} = \hat{a}\hat{\nu}(b)e^t, \quad \hat{a} \in \hat{G}, \quad b \in B = L_+G_0^C, \quad t \in \mathbf{R}. \quad (4.143)$$

Consider first the element $\Pi_0^C(\hat{K}) \in LG_0^C$. It can be uniquely decomposed as

$$\Pi_0^C(\hat{K}) = ab, \quad a \in LG_0, \quad b \in L_+G_0^C. \quad (4.144)$$

Now it is certainly true that \hat{K} can be found among the elements of the \mathbf{C}^\times fiber $(\Pi_0^C)^{-1}(a)\hat{\nu}(b)$ above ab . The existence of \hat{a}, b and t from (4.143) then follows immediately.

It remains to be proved that the decomposition (4.143) is unique. The required argument is very similar to that of the proof of Lemma 4.16 and we shall not repeat it here.

The lemma is proved. #

There exists the pregnant way of presenting the commutator in the Lie algebra $\tilde{\tilde{D}}$. It reads

$$\begin{aligned} & [(X^c, \xi^c(\sigma), x^c), (Y^c, \eta^c(\sigma), y^c)] = \\ & = (0, [\xi^c, \eta^c] - iX^c\partial_\sigma\eta^c + iY^c\partial_\sigma\xi^c, \frac{i}{2\pi} \int (\xi^c, \partial_\sigma\eta^c)_{\mathcal{G}_0^C}). \end{aligned} \quad (4.145)$$

Here X^c, Y^c, x^c, y^c are *complex* numbers and $\xi^c, \eta^c \in LG_0^C$. The identification of various generators is as follows: $\tilde{T}^0 = (i, 0, 0)$ corresponds to the automorphisms S and $\tilde{t}_\infty^1 = (1, 0, 0)$ to the automorphisms Q . Moreover, $\tilde{T}^\infty = (0, 0, i)$ is to be identified with the central circle generator and $\tilde{t}_0^1 = (0, 0, 1)$ with the central line generator corresponding to the group \mathbf{R} in the decomposition $\tilde{B} = \mathbf{R} \times \hat{B}$. We stress that we view $\tilde{\tilde{D}}$ as the real Lie algebra, nevertheless we observe that the commutator just defined is complex bilinear hence $\tilde{\tilde{D}}$ possesses the natural complex structure. Moreover, it is instructive to compare this formula with the commutator (2.11) of the central biextension algebra $\tilde{\mathcal{G}}$. We observe immediately that $\tilde{\mathcal{G}}$ is the real form of $\tilde{\tilde{D}}$ viewed as the complex Lie algebra. In other words, the affine Lu-Weinstein-Soibelman double of $\tilde{\tilde{D}}$ is nothing but the complexification $\tilde{\mathcal{G}}^C$,

in full analogy with the state of matters for the finite dimensional ordinary Lu-Weinstein-Soibelman double. We can define the following invariant non-degenerate bilinear form on $\tilde{\mathcal{G}}^{\mathbf{C}}$

$$\begin{aligned} ((X^c, \xi^c, x^c), (Y^c, \eta^c, y^c))_{\tilde{\mathcal{G}}^{\mathbf{C}}} = \\ (\xi^c, \eta^c)_{\mathcal{G}^{\mathbf{C}}} + X^c y^c + Y^c x^c, \end{aligned} \quad (4.146)$$

where the form $(\cdot, \cdot)_{\mathcal{G}^{\mathbf{C}}}$ was defined in (4.105). The invariant non-degenerate bilinear form on the double $\tilde{\mathcal{D}}$ is then defined as

$$((X^c, \xi^c, x^c), (Y^c, \eta^c, y^c))_{\tilde{\mathcal{D}}} = \frac{1}{\varepsilon} \text{Im}((X^c, \xi^c, x^c), (Y^c, \eta^c, y^c))_{\tilde{\mathcal{G}}^{\mathbf{C}}} \quad (4.147)$$

Note that the form (4.147) is the analogue of the forms (4.106) and (4.81).

Lemma 4.21: The group \tilde{D} is indeed the common Heisenberg double of the groups \tilde{G} and \tilde{B} .

Proof: The parts of this proposition were already proved in the preceding lemmas. The only thing that remains is to prove the isotropy of the Lie algebras $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{B}}$ with respect to the bilinear form (4.147).

We start with $\tilde{\mathcal{G}}$. In the parametrization of $\tilde{\mathcal{D}}$ given by (4.145), we have $\tilde{\mathcal{G}} = \text{Span}(iR, \mathcal{G}, ir)$, $R, r \in \mathbf{R}$ and from (4.146) and (4.147) it follows $(\tilde{\mathcal{G}}, \tilde{\mathcal{G}})_{\tilde{\mathcal{D}}} = 0$; On the other hand, $\tilde{\mathcal{B}} = \text{Span}(R, \mathcal{B}, r)$, $R, r \in \mathbf{R}$ and again $(\tilde{\mathcal{B}}, \tilde{\mathcal{B}})_{\tilde{\mathcal{D}}} = 0$.

The lemma is proved.

#

Conclusion: We have shown that \tilde{B} is the direct product of the central line \mathbf{R} with \hat{B} , i.e. $\tilde{B} = \mathbf{R} \times \hat{B}$ and $\hat{B} = \mathbf{R} \times_Q B$. Thus the requirements of the definition 4.10 of the WZW double \tilde{D} are indeed verified for the affine Lu-Weinstein-Soibelman double and we can safely perform the symplectic reduction leading to the quasitriangular WZW model.

Chapter 5

Loop group quasitriangular WZW model

We are arriving at the core of this paper. We shall consider the loop group quasitriangular WZW model corresponding to the affine Lu-Weinstein-Soibelman double $\tilde{\tilde{D}}$ introduced in the previous chapter. The definition of this model involves certain symplectic reduction (cf. Definition 4.13 of Section 4.3), which is the quasitriangular analogue of the full left-right reduction described in Sections 2.2.2 and 2.2.3. However, in the case of the standard loop group WZW model described in Chapters 2 and 3, we have used also the alternative approach by performing the reduction at the *chiral* level. The full left-right WZW model was then obtained by glueing two copies of the reduced chiral model. Although both approaches gave the same result, the chiral reduction was much simpler from both conceptual and technical points of view.

Here we shall take the advantage of the fact that in the loop group case one can also construct a chiral second floor quasitriangular master model. Therefore we can perform the symplectic reduction directly at the chiral level. As in the standard case, the full left-right quasitriangular WZW model will be then obtained by an appropriate glueing of the reduced chiral copies. It can be shown that this gives the same result as the reduction of the full left-right master model on $\tilde{\tilde{D}}$ in the spirit of the Definition 4.13.

Our strategy here will be therefore as follows: first we introduce the quasitriangular chiral master \tilde{G} -model and perform the quasitriangular analogue of the two-step chiral symplectic reduction of Chapter 3. In this way, we con-

struct the main result of our paper which is the chiral quasitriangular WZW model. Moreover, we shall give the very explicit description of its symplectic structure and of its Hamiltonian making thus evident that we have really to do with the one-parameter deformation of the standard chiral WZW model described in Section 3.2.4. We shall finally glue up two copies of the chiral model to obtain the full left-right quasitriangular WZW theory.

5.1 Quasitriangular chiral geodesical model

We shall first illustrate the idea of the quasitriangular chiral decomposition in the (finite-dimensional) case of the geodesical model on the Lu-Weinstein-Soibelman double $G_0^{\mathbf{C}}$. This section should serve as the ideological and technical reference for the more complicated infinite-dimensional structures described later.

5.1.1 Chiral splitting of the Semenov-Tian-Shansky form

It is the well-known fact [49] that every element K of a simple complex connected and simply connected group $G_0^{\mathbf{C}}$ can be decomposed as

$$K = k_L a k_R^{-1}, \quad k_{L,R} \in G_0, \quad a \in A_+ = \exp \Lambda_0(\mathcal{A}_+^0). \quad (5.1)$$

Here G is the compact real form of $G_0^{\mathbf{C}}$ and $\mathcal{A}_+^0 \subset \Upsilon_0^{-1}(\mathcal{T}) \subset \mathcal{G}_0^*$ is the positive Weyl chamber introduced in Section 3.1.1. Moreover, Λ_0 is the identification map $\Lambda_0 : \mathcal{G}_0^* \rightarrow \mathcal{B}_0$ defined as

$$(\Lambda_0(x^*), y)_{\mathcal{D}} = \langle x^*, y \rangle, \quad x^* \in \mathcal{G}_0^*, y \in \mathcal{G}_0. \quad (5.2)$$

Note that Λ_0 depends on the parameter ε (cf. (4.81)), since the form $(\cdot, \cdot)_{\mathcal{D}}$ does, too. It is easy to see that $A_+ = \exp \Lambda_0(\mathcal{A}_+^0)$ is the subset of the group A appearing in the Iwasawa decomposition $G_0^{\mathbf{C}} = G_0 A N$. This subset *does not* depend on ε , however, since the Weyl chamber is invariant under the scaling.

The decomposition (5.1) is called the Cartan one and its ambiguity is given by the simultaneous right multiplication of k_L and k_R by the same element of \mathbf{T} . Note that the same thing was true for the Cartan decomposition of T^*G_0 (cf. Theorem 3.2).

Consider the manifold $G_0 \times A_+ \times G_0$; we shall denote its points as triples (k_L, a, k_R) . The Cartan decomposition (5.1) then induces a natural map Ξ from this manifold into the complex Heisenberg double $D = G_0^{\mathbb{C}}$. We can then pull-back the Semenov-Tian-Shansky symplectic form ω by the map Ξ . The following lemma is of a crucial importance for the success of all our programme:

Lemma 5.1: Consider maps $\Xi_{L,R} : G_0 \times A_+ \times G_0 \rightarrow D$ defined as $\Xi_L(k_L, a, k_R) = k_L a$ and $\Xi_R(k_L, a, k_R) = a k_R^{-1}$. Then

$$\Xi^* \omega = \Xi_L^* \omega + \Xi_R^* \omega. \quad (5.3)$$

Remark: The proposition of the lemma 5.1 can be restated intuitively as follows: The pullback form $\Xi^* \omega$ can be chirally decomposed on the left and right part who talk to each other only via the variable a .

Proof:

First we write the Semenov-Tian-Shansky form ω as follows

$$\omega = -\frac{1}{2}(g_R^{-1} dg_R \frown K^{-1} dK)_{\mathcal{D}} - \frac{1}{2}(dg_L g_L^{-1} \frown dK K^{-1})_{\mathcal{D}}. \quad (5.4)$$

Recall (cf. (4.11)) that $K^{-1} dK$ denotes the left invariant Maurer-Cartan form on the Heisenberg double D and $g_R^{-1} dg_R$ is $g_R^* \lambda_{G_0}$, where g_R and g_L are induced by the decomposition

$$K = b_L(K) g_R(K) = g_L(K) b_R(K). \quad (5.5)$$

We can easily recover the original formula (4.8) for ω by using the isotropy of \mathcal{G}_0 with respect to $(.,.)_{\mathcal{D}}$ and by noting that

$$K^{-1} dK = g_R^{-1} (b_L^{-1} db_L) g_R + g_R^{-1} dg_R, \quad dK K^{-1} = dg_L g_L^{-1} + g_L (db_R b_R^{-1}) g_L^{-1}. \quad (5.6)$$

Consider now another decomposition of $K \in D$:

$$K = p_L k = k p_R, \quad (5.7)$$

where

$$k = k_L k_R^{-1}, \quad p_L = k_L a k_L^{-1}, \quad p_R = k_R a k_R^{-1}. \quad (5.8)$$

Clearly, $k_{L,R}$ and a come from the Cartan decomposition (5.1). Although they are not given unambiguously, this ambiguity disappears in (5.8); in other words: k and $p_{L,R}$ are uniquely fixed by $K \in D$. We can successively rewrite the form ω as follows

$$\begin{aligned}
-2\omega &= (g_R^{-1}dg_R \frown K^{-1}dK)_{\mathcal{D}} + (dg_Lg_L^{-1} \frown dKK^{-1})_{\mathcal{D}} = \\
&= (g_R^{-1}dg_R \frown k^{-1}p_L^{-1}dp_Lk)_{\mathcal{D}} + (dg_Lg_L^{-1} \frown kdp_Rp_R^{-1}k^{-1})_{\mathcal{D}} = \\
&= (kg_R^{-1}d(g_Rk^{-1}) \frown p_L^{-1}dp_L)_{\mathcal{D}} + (d(k^{-1}g_L)g_L^{-1}k \frown dp_Rp_R^{-1})_{\mathcal{D}} + \\
&\quad + (dkk^{-1} \frown p_L^{-1}dp_L)_{\mathcal{D}} + (k^{-1}dk \frown dp_Rp_R^{-1})_{\mathcal{D}}. \tag{5.9}
\end{aligned}$$

In deriving this equality, we have used the isotropy of \mathcal{G}_0 with respect to $(\cdot, \cdot)_{\mathcal{D}}$. Now we see from (5.5) and (5.7) that

$$g_Rk^{-1} = b_L^{-1}p_L, \quad k^{-1}g_L = p_Rb_R^{-1}. \tag{5.10}$$

This permits to write

$$\begin{aligned}
-2\omega &= (dp_Lp_L^{-1} \frown db_Lb_L^{-1})_{\mathcal{D}} + (p_R^{-1}dp_R \frown b_R^{-1}db_R)_{\mathcal{D}} + \\
&\quad + (dkk^{-1} \frown p_L^{-1}dp_L)_{\mathcal{D}} + (k^{-1}dk \frown dp_Rp_R^{-1})_{\mathcal{D}}. \tag{5.11}
\end{aligned}$$

Now there are four terms on the r.h.s. of (5.11). We insert the expressions (5.8) into the last two of them. The result reads

$$\begin{aligned}
\Xi^*\omega &= \frac{1}{2}(db_Lb_L^{-1} \frown dp_Lp_L^{-1})_{\mathcal{D}} + \frac{1}{2}(b_R^{-1}db_R \frown p_R^{-1}dp_R)_{\mathcal{D}} + \\
&\quad \frac{1}{2}((daa^{-1} + a^{-1}da) \frown k_L^{-1}dk_L)_{\mathcal{D}} - \frac{1}{2}((daa^{-1} + a^{-1}da) \frown k_R^{-1}dk_R)_{\mathcal{D}} + \\
&\quad + \frac{1}{2}(k_L^{-1}dk_L \frown ak_L^{-1}dk_La^{-1})_{\mathcal{D}} - \frac{1}{2}(k_R^{-1}dk_R \frown ak_R^{-1}dk_Ra^{-1})_{\mathcal{D}}. \tag{5.12}
\end{aligned}$$

Now observe that

$$b_L(k_Lak_R^{-1}) = b_L(k_La), \quad b_R(k_Lak_R^{-1}) = b_R(ak_R^{-1}). \tag{5.13}$$

This means that quantities bearing the index L (R) do not depend on k_R (k_L) hence the proposition of the lemma follows.

#

Consider now the model spaces $M_L = G_0 \times \mathcal{A}_+^0$ and $M_R = G_0 \times \mathcal{A}_-^0$, where $\mathcal{A}_-^0 = -\mathcal{A}_+^0$.

The symplectic form ω_L on M_L is defined as

$$\begin{aligned}\omega_L &= \tilde{\Xi}_L^* \omega = \\ &= \frac{1}{2}(db_L b_L^{-1} \frown dp_L p_L^{-1})_{\mathcal{D}} + (daa^{-1} \frown k_L^{-1} dk_L)_{\mathcal{D}} + \frac{1}{2}(k_L^{-1} dk_L \frown ak_L^{-1} dk_L a^{-1})_{\mathcal{D}}.\end{aligned}\tag{5.14}$$

The map $\tilde{\Xi}_L : M_L \rightarrow D$ is given by $\tilde{\Xi}(k_L, \phi_L) = k_L a_L$ for $(k_L, \phi_L) \in M_L$, where

$$a_L = \exp \Lambda_0(\phi_L)\tag{5.15}$$

and p_L and b_L are defined as before.

Now a subtlety: we define the symplectic form ω_R on M_R by exactly the same formula, i.e.,

$$\omega_R = \tilde{\Xi}_R^* \omega,\tag{5.16}$$

where $\tilde{\Xi}_R(k_R, \phi_R) = k_R a_R$, where $a_R = \exp \Lambda_0(\phi_R)$. Such a definition may look surprising because the right part of the form $\Xi^* \omega$ in (5.12) was obtained by pulling back by the map $a_L k_R^{-1}$ rather than by $k_R a_R$. We shall see in a while that this gives the same thing, however. The advantage of our definition is evident: it allows us to study only the case M_L since the symplectic structure on M_R is the same (up to the change in the domain of the variable $\phi: \mathcal{A}_+^0 \rightarrow \mathcal{A}_-^0$).

Consider the manifold $M_L \times M_R$ equipped with the symplectic form

$$\omega_{L \times R} = \omega_L + \omega_R.\tag{5.17}$$

Lemma 5.2: The submanifold of $M_L \times M_R$, given by equating $a_L a_R = 1$, is naturally diffeomorphic to $G_0 \times A_+ \times G_0$ and the form $\omega_{L \times R}$ restricted to this submanifold is nothing but $\Xi^* \omega$ given by the equation (5.12).

Proof: The left part of $\Xi^* \omega$ (cf. (5.12)) coincides with ω_L by definition. Let us show that the right part gives ω_R . We have from (5.14) and (5.16)

$$\omega_R = \frac{1}{2}(db_L(k_R a_R) b_L^{-1}(k_R a_R) \frown dp_R p_R^{-1})_{\mathcal{D}} +$$

$$+ (da_R a_R^{-1} \frown k_R^{-1} dk_R)_{\mathcal{D}} + \frac{1}{2} (k_R^{-1} dk_R \frown a_R k_R^{-1} dk_R a_R^{-1})_{\mathcal{D}}, \quad (5.18)$$

where $p_R = k_R a_R k_R^{-1}$. Now we have to set $a_R = a_L^{-1} \equiv a^{-1}$ and insert in (5.18). We obtain

$$\begin{aligned} \omega_R = & \frac{1}{2} (db_L(k_R a^{-1}) b_L^{-1}(k_R a^{-1}) \frown d(k_R a^{-1} k_R^{-1}) k_R a k_R^{-1})_{\mathcal{D}} + \\ & + (k_R^{-1} dk_R \frown da a^{-1})_{\mathcal{D}} - \frac{1}{2} (k_R^{-1} dk_R \frown a k_R^{-1} dk_R a^{-1})_{\mathcal{D}}. \end{aligned} \quad (5.19)$$

Now we see that the second and third term on the r.h.s. of (5.19) have their counterparts in the right part of $\Xi^* \omega$. It remains to show that

$$\begin{aligned} & (d(k_R a^{-1} k_R^{-1}) k_R a k_R^{-1} \frown db_L(k_R a^{-1}) b_L^{-1}(k_R a^{-1}))_{\mathcal{D}} = \\ & = ((k_R a^{-1} k_R^{-1}) d(k_R a k_R^{-1}) \frown b_R^{-1}(a k_R^{-1}) db_R(a k_R^{-1}))_{\mathcal{D}}. \end{aligned} \quad (5.20)$$

But this equality follows from the following obvious relation

$$b_L(k_R a^{-1}) = b_R^{-1}(a k_R^{-1}). \quad (5.21)$$

The lemma is proved. #

Corollary 5.3: The symplectic reduction of the form $\omega_{L \times R}$ by the relation $a_L a_R = 1$ is the Semenov-Tian-Shansky form ω on $G_0^{\mathbf{C}}$.

Proof: The form $\omega_{L \times R}$ restricted on $a_L a_R = 1$ coincides with $\Xi^* \omega$ given by (5.12) and it is clearly degenerate along orbits of the maximal torus \mathbf{T} acting as $(a, k_L, k_R) \rightarrow (a, k_L h, k_R h)$, $h \in \mathbf{T}$. The orbit space is nothing but $G_0^{\mathbf{C}}$. #

So far we have shown that the symplectic structure ω on $D = G_0^{\mathbf{C}}$ naturally originates from $\omega_{L \times R}$ under the symplectic reduction induced by setting $a_L a_R = 1$. But also the Hamiltonian $H(K)$ introduced in Section 4.2 can be "descended" from some Hamiltonian on $M_L \times M_R$. Indeed, the latter is defined as follows

$$H_{L \times R} = H_L + H_R = -\frac{1}{2} (\phi_L, \phi_L)_{\mathcal{G}_0^*} - \frac{1}{2} (\phi_R, \phi_R)_{\mathcal{G}_0^*} = \frac{1}{2} a_L^\mu a_L^\mu + \frac{1}{2} a_R^\mu a_R^\mu. \quad (5.22)$$

Recall that

$$a_L = \exp(a_L^\mu \Lambda_0(t_\mu)), \quad a_R = \exp(a_R^\mu \Lambda_0(t_\mu)). \quad (5.23)$$

Clearly, $\Lambda_0(t_\mu)$'s are in $Lie(A) \subset \mathcal{B}_0$, they fulfil $(\Lambda_0(t_\mu), T^\nu)_{\mathcal{D}} = \langle t_\mu, T^\mu \rangle = \delta_\mu^\nu$ and $T^\mu \in \mathcal{G}_0$ were defined in (3.24). By the abuse of notation, we shall often write $\Lambda_0(t_i) = t_i$, i.e; the identification map will be tacitly assumed (cf. (4.6)). However, in the case where a confusion can arise we shall use the symbol Λ_0 explicitly.

Note that the Hamiltonians $H_{L,R}$ on $M_{L,R}$ coincide with the Hamiltonians of the standard geodesical model (3.12). It is the symplectic form ω_L , defined by (5.14), that differs from $d\theta_L$ given by (3.11). We shall see, however, that for $\varepsilon \rightarrow 0$ it holds $\omega_L \rightarrow d\theta_L$. The Hamiltonians do not depend on k_L, k_R , respectively, hence they trivially survive the symplectic reduction and give the Hamiltonian (4.102) on $D = G_0^{\mathbf{C}}$.

Definition 5.4: The chiral quasitriangular geodesical model is the dynamical system whose phase space is M_L parametrized by couples $(k \in G_0, \phi \in \mathcal{A}_+^0)$, whose Hamiltonian is

$$H_L = -\frac{1}{2}(\phi, \phi)_{\mathcal{G}_0^*} = \frac{1}{2}a^\mu a^\mu \quad (5.24)$$

and whose symplectic form is

$$\begin{aligned} \omega_L &= \\ &= \frac{1}{2}(db_L(ka)b_L^{-1}(ka) \frown dpp^{-1})_{\mathcal{D}} + (daa^{-1} \frown k^{-1}dk)_{\mathcal{D}} + \frac{1}{2}(k^{-1}dk \frown a(k^{-1}dk)a^{-1})_{\mathcal{D}}, \end{aligned} \quad (5.25)$$

where $p = kak^{-1}$ and $a = \exp \Lambda_0(\phi) = \exp(a^\mu \Lambda_0(t_\mu))$.

We have learned in this section that the quasitriangular geodesical model formulated on the Lu-Weinstein-Soibelman double $G_0^{\mathbf{C}}$ admits the chiral decomposition into two chiral models defined above. By this we mean that it can be defined by the symplectic reduction of the model defined on $M_L \times M_R$ and characterized by the symplectic form $\omega_{L \times R}$ and the Hamiltonian $H_{L \times R}$.

Remark: The chiral quasitriangular geodesical model was apparently first proposed in [4]. In what follows, we shall invert the symplectic form ω_L in the way which technically differs from that of [4]. Its spirit is the same, however, in that we use the Poisson-Lie symmetry of ω_L .

5.1.2 The power of the Poisson-Lie symmetry

In the paragraphs 5.1.2 and 5.1.3, we shall often parametrize the points of M_L by couples (k, a) , where $k \in G_0$ and $a = e^{\Lambda_0(\phi)} \in A_+$.

Lemma 5.5: The chiral quasitriangular geodesical model is Poisson-Lie symmetric (cf. Definition 4.7) with respect to the left action $(k, a) \rightarrow (k_0 k, a)$ of the group G_0 on the model space M_L . The corresponding non-Abelian moment map $M : M_L \rightarrow B$ is given by $M(k, a) = b_L(ka) = \text{Dres}_k a$.

Proof: Consider a point (k, a) in M_L . This point can be mapped into D as $\tilde{\Xi}(k, a) = ka$ under the embedding $M_L \hookrightarrow D$. Multiplication of (k, a) on the left by an infinitesimal generator $T \in \mathcal{G}_0$ gives the vector $v_T = (Tk, a)$ hence the vector $\tilde{\Xi}_* v_T = Tka = R_{(ka)*} T \in T_{(ka)} D$. We want to show that

$$\omega_L(., v_T) = (T, dMM^{-1})_{\mathcal{D}}. \quad (5.26)$$

Since the form ω_L is the pullback $\tilde{\Xi}^* \omega$, we infer

$$\omega_L(u, v_T) = \omega(\tilde{\Xi}_* u, \tilde{\Xi}_* v_T) = \omega(\tilde{\Xi}_* u, R_{(ka)*} T), \quad (5.27)$$

where u is an arbitrary vector at the point $(k, a) \in M_L$. Now from (4.88), we conclude

$$\omega(\tilde{\Xi}_* u, R_{(ka)*} T) = (T, \langle b_L^* \rho_B, \tilde{\Xi}_* u \rangle)_{\mathcal{D}} = (T, \langle M^* \rho_B, u \rangle)_{\mathcal{D}} = (T, \langle dMM^{-1}, u \rangle)_{\mathcal{D}}, \quad (5.28)$$

because $\tilde{\Xi}^* b_L = M$. The Hamiltonian H_L does not depend on k , it is therefore clearly invariant with respect to $(k, a) \rightarrow (k_0 k, a)$.

The lemma is proved. #

The existence of the non-Abelian moment map $M : M_L \rightarrow B$ entails certain differential condition on the Poisson bivector Σ_L , corresponding to the form ω_L . Recall first, that

$$\Sigma_L(., \omega_L(., u)) = u, \quad (5.29)$$

for arbitrary vector u at arbitrary point of M_L . It then follows from (5.26), that

$$\Sigma_L(., dMM^{-1}) = v, \quad (5.30)$$

where $v \in TM_L \otimes \mathcal{G}_0^* (\equiv TM_L \otimes \mathcal{B}_0)$ generates the left action of G_0 on the model space M_L .

Calculate now the Lie derivative $\mathcal{L}_v \Sigma_L$ with respect to v . The result is

$$\begin{aligned} \mathcal{L}_v \Sigma_L &= \mathcal{L}_v \omega_L^{-1} = -\Sigma_L \mathcal{L}_v \omega_L \Sigma_L = -\Sigma_L d i_v \omega_L \Sigma_L = \\ &+ \Sigma_L d(d M M^{-1}) \Sigma_L = \Sigma_L (d M M^{-1} \wedge d M M^{-1}) \Sigma_L = -v \wedge v. \end{aligned} \quad (5.31)$$

The last relation can be also written in some basis T^i of \mathcal{G}_0 :

$$\mathcal{L}_{v^i} \Sigma_L = -\frac{1}{2} \tilde{f}_{jk}^i v^j \wedge v^k, \quad (5.32)$$

where

$$\tilde{f}_{jk}^i = (T^i, [t_j, t_k])_{\mathcal{D}}, \quad (t_i, T^j)_{\mathcal{D}} = \delta_i^j, \quad v^i = (v, T^i)_{\mathcal{D}}. \quad (5.33)$$

This is the differential condition that the Poisson tensor Σ_L must obey.

Note first that there is a particular solution of the differential condition (5.32). It reads

$$\Sigma_L^{part}(k, a) = -\frac{1}{2} \Pi_{ij}^R(k) R_{k*} T^i \wedge R_{k*} T^j, \quad (5.34)$$

where the matrix $\Pi_{ij}^R(k)$ is given by

$$\Pi_{ij}^R(k) = -B_{ik}(k) A_j^k(k^{-1}), \quad (5.35)$$

$$B_{ik}(k) = (k^{-1} t_i k, t_k)_{\mathcal{D}}, \quad A_j^k(k) = (k^{-1} T^k k, t_j)_{\mathcal{D}}. \quad (5.36)$$

Now we replace the role of the groups G and B in the Proposition 4.5 in order to realize that, up to the sign minus, the r.h.s. of (5.35) is the Poisson-Lie bivector on the group manifold G_0 (here viewed as the bivector on $G_0 \times A_+$). If we regard the condition (5.32) as the differential equation for the unknown bivector Σ_L , we see immediately that its general solution can be given as the sum of the particular solution (5.34) and any solution of the homogeneous equation. But the latter is nothing but every G_0 -right-invariant bivector because $v^i = R_{k*} T^i$. We can therefore write the following ansatz for the seeked Poisson bivector Σ^L

$$\begin{aligned} \Sigma_L(k, a) &= -\frac{1}{2} \Pi_{ij}^R(k) R_{k*} T^i \wedge R_{k*} T^j + \\ &+ \Sigma_{ij}^0(a) L_{k*} T^i \wedge L_{k*} T^j + \sigma_i^\mu(a) L_{k*} T^i \wedge \frac{\partial}{\partial a^\mu} + s^{\mu\nu}(a) \frac{\partial}{\partial a^\mu} \wedge \frac{\partial}{\partial a^\nu} \end{aligned} \quad (5.37)$$

Our task is to find the coefficient functions $\Sigma_{ij}^0(a)$, $\sigma_i^\mu(a)$ and $s^{\mu\nu}(a)$.

Thus we observe the power of the Poisson-Lie symmetry: the Poisson tensor Σ_L on the model space M_L is completely determined by its value at the points $(e, a) \in M_L$, in other words: at the unit element e of G_0 .

5.1.3 Deformed dynamical r -matrix

Here we shall evaluate the unknown functions $\Sigma_{ij}^0(a)$, $\sigma_i^\mu(a)$ and $s^{\mu\nu}(a)$. For this, we first calculate the matrix of the symplectic form ω_L at points (e, a) (in the basis $R_{a*}T^i, R_{a*}t_\mu$ of $T_{(e,a)}M_L$) and then we invert it to obtain the unknown coefficient functions of the Poisson bracket.

The basis T^i of \mathcal{G}_0 was canonically chosen in (3.24), i.e. $T^i = (T^\mu, B^\alpha, C^\alpha)$, where

$$T^\mu = iH^\mu, \quad B^\alpha = \frac{i}{\sqrt{2}}(E^\alpha + E^{-\alpha}), \quad C^\alpha = \frac{1}{\sqrt{2}}(E^\alpha - E^{-\alpha}), \quad (3.24).$$

Recall then that a is an element of the group A ; its Lie algebra $Lie(A)$ is generated by the generators $t_\mu = \varepsilon H^\mu$ which are dual to iH^μ with respect to the bilinear form $(\cdot, \cdot)_{\mathcal{D}}$ given by the formula (4.81). The full dual basis reads

$$t_i = (t_\mu, b_\alpha, c_\alpha) = (\varepsilon H^\mu, \frac{\varepsilon|\alpha|^2}{\sqrt{2}}E^\alpha, -i\frac{\varepsilon|\alpha|^2}{\sqrt{2}}E^\alpha) \quad (5.38)$$

and it satisfies the basic condition $(t_i, T^j)_{\mathcal{D}} = \delta_i^j$.

Remark: Note that the form $(\cdot, \cdot)_{\mathcal{D}}$ given by (4.81) depends on ε therefore also the dual generators t_i are ε -dependent. We shall occasionally stress this dependance by writing $(\cdot, \cdot)_\varepsilon$ and t_i^ε .

We decompose the elements of the basis of $T_{(e,a)}M_L$ into several parts: the α -part generated by $R_{a*}B^\alpha, R_{a*}C^\alpha$, $\alpha \in \Phi_+$ and the μ -part generated by $R_{a*}T^\mu, R_{a*}t_\mu$. Now we use the formula (4.18)

$$\omega(t, u) = (t, (\Pi_{\tilde{L}R} - \Pi_{L\tilde{R}})u)_{\mathcal{D}}.$$

We immediately observe that

$$\omega_L(\alpha, \mu) = 0, \quad \omega_L(\alpha, \beta) = 0, \quad \alpha, \beta \in \Phi_+, \quad \alpha \neq \beta. \quad (5.39)$$

This is because we are at the point $(e, a) \in M_L(\subset D)$. It is hence sufficient to invert the matrix ω_L for the μ -sector and for every α -sector separately. We have

$$\Pi_{L\tilde{R}}R_{a*}T^\mu = 0, \quad \Pi_{\tilde{L}R}R_{a*}T^\mu = R_{a*}T^\mu. \quad (5.40)$$

From this we obtain

$$\omega_L(R_{a*}t_\nu, R_{a*}T^\mu) = \delta_\nu^\mu. \quad (5.41)$$

One has

$$R_{a*}C^\alpha = (R_{a*}C^\alpha, L_{a*}t_i)_{\mathcal{D}} L_{a*}T^i + (R_{a*}C^\alpha, L_{a*}T^i)_{\mathcal{D}} L_{a*}t_i \quad (5.42)$$

and from (5.38)

$$L_{a*}b_\alpha = e^{a^\mu \langle \alpha, t_\mu \rangle} R_{a*}b_\alpha, \quad L_{a*}c_\alpha = e^{a^\mu \langle \alpha, t_\mu \rangle} R_{a*}c_\alpha. \quad (5.43)$$

Then we have

$$\Pi_{L\tilde{R}} R_{a*}C^\alpha = e^{a^\mu \langle \alpha, t_\mu \rangle} (L_{a*}B^\alpha, R_{a*}C^\alpha)_{\mathcal{D}} R_{a*}b_\alpha + e^{a^\mu \langle \alpha, t_\mu \rangle} (L_{a*}C^\alpha, R_{a*}C^\alpha)_{\mathcal{D}} R_{a*}c_\alpha; \quad (5.44)$$

$$\Pi_{\tilde{L}R} R_{a*}C^\alpha = R_{a*}C^\alpha. \quad (5.45)$$

It follows that

$$\omega_L(R_{a*}B^\alpha, R_{a*}C^\alpha) = -e^{a^\mu \langle \alpha, t_\mu \rangle} (L_{a*}B^\alpha, R_{a*}C^\alpha)_{\mathcal{D}} = \frac{1}{\varepsilon|\alpha|^2} (e^{2a^\mu \langle \alpha, t_\mu \rangle} - 1). \quad (5.46)$$

Our conclusion is that

$$s^{\mu\nu} = 0, \quad \sigma_\alpha^\mu = 0, \quad \sigma_\mu^\nu = \delta_\mu^\nu, \quad \Sigma_{\mu\nu}^0 = \Sigma_{\alpha\mu}^0 = \Sigma_{\alpha\beta}^0 = 0 \quad (5.47)$$

and

$$\begin{aligned} \Sigma_L(k, a) &= -\frac{1}{2} \Pi_{ij}^R(k) R_{k*}T^i \wedge R_{k*}T^j + \\ &+ \sum_{\alpha \in \Phi_+} \frac{\varepsilon|\alpha|^2}{(1 - e^{2\varepsilon a^\mu \langle \alpha, H^\mu \rangle})} L_{k*}B^\alpha \wedge L_{k*}C^\alpha + L_{k*}T^\mu \wedge \frac{\partial}{\partial a^\mu}. \end{aligned} \quad (5.48)$$

Recall that in the formulas above, we have set $a = e^{a^\mu t_\mu}$ and $t_\mu = \varepsilon H^\mu$.

Denote $\{.,.\}_{qM_L}$ the Poisson bracket corresponding to the bivector Σ_L . Now we wish to calculate the bracket of the type $\{k \otimes k\}_{qM_L}$; according to (5.48), it can be decomposed in two parts

$$\{k \otimes k\}_{qM_L} = \{k \otimes k\}_{\Sigma^0} - \{k \otimes k\}_{G_0}^R, \quad (5.49)$$

where, of course, k is understood in some representation ρ , the bracket $\{k \otimes k\}_{\Sigma^0}$ is associated to the bivector on the second line of (5.48) and the bracket

$\{k \otimes k\}_{G_0}^R$ is the standard Poisson-Lie bracket (4.70) on the group G_0 . Let us calculate the latter more explicitly :

$$\{k \otimes k\}_{G_0}^R = (T^i k \otimes T^j k) \Pi_{ij}^R(k) = -(T^i k \otimes T^j k)(k^{-1} t_i k, t_l)_{\mathcal{D}} (k T^l k^{-1}, t_j)_{\mathcal{D}}. \quad (5.50)$$

From the isotropy of \mathcal{G}_0 , it follows

$$(k T^l k^{-1}, t_j)_{\mathcal{D}} T^j = k T^l k^{-1}.$$

Inserting this back into (5.50), we obtain

$$\{k \otimes k\}_{G_0}^R = -(T^i k \otimes k T^l)(k^{-1} t_i k, t_l)_{\mathcal{D}}. \quad (5.51)$$

We insert in (5.51) another obvious identity:

$$k^{-1} t_i k = (k^{-1} t_i k, T^l)_{\mathcal{D}} t_l + (k^{-1} t_i k, t_l)_{\mathcal{D}} T^l$$

to obtain finally

$$\{k \otimes k\}_{G_0}^R = -T^i k \otimes t_i k + k T^i \otimes k t_i = [(k \otimes k), (T^i \otimes t_i)]. \quad (5.52)$$

We recall that $D = G_0^{\mathbb{C}}$ was viewed as the *real* group. Among the representations of \mathcal{D} , we can therefore consider those originating from the complex representations of $\mathcal{G}_0^{\mathbb{C}}$. From now on we restrict our attention to the faithful representations of this type. The representatives of the elements $B^\alpha, C^\alpha, b_\alpha, c_\alpha$ are then obtained from the representatives of E^α by using the formulae (3.24) and (5.38). In such representations and using the canonical choice of the basis (3.24) and (5.38), we can calculate

$$r_+ \equiv T^i \otimes t_i = i\varepsilon(H^\mu \otimes H^\mu + \sum_{\alpha>0} |\alpha|^2 E^{-\alpha} \otimes E^\alpha) = i\varepsilon C + \varepsilon r. \quad (5.53)$$

Here

$$C \equiv H^\mu \otimes H^\mu + \sum_{\alpha \in \Phi_+} \frac{|\alpha|^2}{2} (E^\alpha \otimes E^{-\alpha} + E^{-\alpha} \otimes E^\alpha) \quad (5.54)$$

is the Casimir element and

$$r \equiv \sum_{\alpha \in \Phi_+} \frac{i|\alpha|^2}{2} (E^{-\alpha} \otimes E^\alpha - E^\alpha \otimes E^{-\alpha}) \quad (5.55)$$

is the so-called classical r -matrix. We note that the Casimir element commutes with the diagonal elements like $(k \otimes k)$. Hence, we can rewrite (5.52) as

$$-\{k \otimes k\}_{G_0}^R = \varepsilon[r, (k \otimes k)]. \quad (5.56)$$

Now we use (5.56) and (5.48) to write down the Poisson bracket:

$$\{k \otimes k\}_{M_L} = (k \otimes k)r'_\varepsilon(a^\mu) - \{k \otimes k\}_{G_0}^R = (k \otimes k)r'_\varepsilon(a^\mu) + \varepsilon[r, (k \otimes k)], \quad (5.57)$$

where

$$r'_\varepsilon(a^\mu) = \sum_{\alpha \in \Phi_+} \frac{\varepsilon|\alpha|^2}{(1 - e^{2\varepsilon a^\mu \langle \alpha, H^\mu \rangle})} (B^\alpha \otimes C^\alpha - C^\alpha \otimes B^\alpha). \quad (5.58)$$

The Poisson bracket (5.57) can be rewritten as

$$\{k \otimes k\}_{qM_L} = (k \otimes k)r_\varepsilon(a^\mu) + \varepsilon r(k \otimes k), \quad (5.59)$$

where $r_\varepsilon(a^\mu)$ is the so called canonical dynamical r -matrix associated to a simple Lie algebra (see e.g. [48]). It is given by

$$r_\varepsilon(a^\mu) = r'_\varepsilon(a^\mu) - \varepsilon r = i\varepsilon \sum_{\alpha \in \Phi} \frac{|\alpha|^2}{2} \coth(\varepsilon a^\mu \langle \alpha, H^\mu \rangle) E^\alpha \otimes E^{-\alpha}. \quad (5.60)$$

It is interesting to observe that the deformed braiding relation (5.59) involves two canonical r -matrices: the standard one and the dynamical one. The description of the full Poisson bivector Σ_L on M_L is then completed by the following bracket

$$\{k, a^\mu\}_{qM_L} = kT^\mu. \quad (5.61)$$

It is important to calculate the limit $\varepsilon \rightarrow 0$ (or, equivalently, $q \rightarrow 1$) of the dynamical r -matrix $r_\varepsilon(a^\mu)$ and of the Poisson brackets $\{k \otimes k\}_{qM_L}$, $\{k, a^\mu\}_{qM_L}$. Recall the explicit expression for the dynamical r -matrix $r_0(a^\mu)$ obtained in (3.30):

$$r_0(a^\mu) = \sum_{\alpha \in \Phi_+} \frac{i|\alpha|^2}{2a^\mu \langle \alpha, H^\mu \rangle} E^\alpha \otimes E^{-\alpha}. \quad (5.62)$$

We observe immediately that $\lim_{\varepsilon \rightarrow 0} r_\varepsilon(a^\mu) = r_0(a^\mu)$. Looking at (3.34), (3.35), (5.59) and (5.61), we conclude that $\lim_{q \rightarrow 1} \{.,.\}_{qM_L} = \{.,.\}_{M_L}$. In other words: the symplectic structure of the chiral quasitriangular geodesical model

is the smooth q -deformation of the symplectic structure of the standard chiral geodesical model. The same conclusion can be obviously obtained also by studying directly the $\varepsilon \rightarrow 0$ limit of the bivector (5.48).

Our next task is to calculate the Poisson bracket $\{b_L(ka) \otimes b_L(ka)\}_{qM_L}$ of the non-Abelian moment maps $b_L(ka)$. The simplest way to do it is to use Lemma 5.5 and realize that for any function $f(k, a)$ on M_L it holds

$$(T^i, \{f(k, a), b_L(ka)\}_{qM_L} b_L^{-1}(ka))_{\mathcal{D}} = \langle \nabla_{G_0}^L f, T^i \rangle \quad (5.63)$$

or, equivalently,

$$\{f(k, a), b_L(ka)\}_{qM_L} = \langle \nabla_{G_0}^L f, T^i \rangle t_i b_L(ka). \quad (5.64)$$

In particular, we have

$$\{k \otimes b_L(ka)\}_{qM_L} = (T^i \otimes t_i)(k \otimes b_L(ka)) = r_+(k \otimes b_L(ka)). \quad (5.65)$$

Replacing $f(k, a)$ by the matrix valued function $b_L(ka)$, we obtain

$$\begin{aligned} \{b_L(ka) \otimes b_L(ka)\}_{M_L} &= b_L(T^i ka) \otimes t_i b_L(ka) = \\ &= (b_L^{-1} T^i b_L, T^j)_{\mathcal{D}} b_L t_j \otimes t_i b_L = \\ &= (T^i \otimes t_i)(b_L \otimes b_L) - (b_L \otimes b_L)(T^i \otimes t_i), \end{aligned} \quad (5.66)$$

where the last equality follows from

$$b_L^{-1} T^i b_L = (b_L^{-1} T^i b_L, T^j)_{\mathcal{D}} t_j + (b_L^{-1} T^i b_L, t_j)_{\mathcal{D}} T^j. \quad (5.67)$$

Using (5.53), we can finally write

$$\{b_L(ka) \otimes b_L(ka)\}_{qM_L} = \varepsilon[r, b_L(ka) \otimes b_L(ka)]. \quad (5.68)$$

In the case of the compact group G_0 , the Poisson brackets of the type $\{k \otimes k\}$ determines completely the Poisson tensor. In the case of the group AN this is no longer true, because (5.68) computes only the Poisson brackets of the holomorphic functions of the variables v_α in the formula (4.83). It turns out that knowing two other matrix Poisson brackets of the form $\{b_L^\dagger(ka) \otimes b_L^\dagger(ka)\}_{qM_L}$ and $\{(b_L^\dagger(ka))^{-1} \otimes b_L(ka)\}_{qM_L}$ is already sufficient. The former bracket can be calculated similarly as before with the result

$$\{b_L^\dagger(ka) \otimes b_L^\dagger(ka)\}_{qM_L} = -\varepsilon[r, b_L^\dagger(ka) \otimes b_L^\dagger(ka)]. \quad (5.69)$$

The calculation of the latter goes as follows

$$\begin{aligned}
& \{(b_L^\dagger(ka))^{-1} \otimes b_L(ka)\}_{q_{M_L}} = -(b_L^{-1}T^i b_L, T^j)_{\mathcal{D}}(b_L^\dagger)^{-1}t_j^\dagger \otimes t_i b_L = \\
& = T^i(b_L^\dagger)^{-1} \otimes t_i b_L - (b_L^{-1}T^i b_L, t_j)_{\mathcal{D}}(b_L^\dagger)^{-1}T^j \otimes t_i b_L = \\
& = (T^i \otimes t_i)((b_L^\dagger)^{-1} \otimes b_L) - ((b_L^\dagger)^{-1} \otimes b_L)(T^i \otimes t_i) = [r_+, ((b_L^\dagger)^{-1} \otimes b_L)]. \quad (5.70)
\end{aligned}$$

In the derivation of the relation above, we have used the following formula

$$-b_L^\dagger T^i (b_L^\dagger)^{-1} = (b_L^{-1}T^i b_L, T^j)_{\mathcal{D}} t_j^\dagger - (b_L^{-1}T^i b_L, t_j)_{\mathcal{D}} T^j, \quad (5.71)$$

which is the consequence of (5.67). The Poisson brackets (5.68), (5.69) and (5.70) constitute the quasitriangular generalization of the commutation relation (3.41) between the coefficients of the standard Abelian moment map $M(k, a) = \beta_L(ka)$ (cf. Section 3.1.3). To see this we compute the $\varepsilon \rightarrow 0$ limit of (5.68), (5.69) and (5.70). This calculation requires some notational care (cf. Remark after Eq. (5.23)). Until the end of this paragraph, we shall make the notational distinction between \mathcal{G}_0^* and \mathcal{B}_0 ; i.e. t_i 's will be always the elements of \mathcal{G}_0^* dual to $T^i \in \mathcal{G}_0$ with respect to the pairing $\langle \cdot, \cdot \rangle$ and $t_i^\varepsilon = \Lambda_0(t_i)$ the elements of \mathcal{B}_0 dual to $T^i \in \mathcal{G}_0$ with respect to the pairing $(\cdot, \cdot)_\varepsilon$.

We start from the formula

$$b(ka) = \text{Dres}_k(a) = 1 + a^\mu C_\mu^j(k) t_j^\varepsilon + O(\varepsilon^2) \quad (5.72)$$

written is some matrix representation of $G_0^{\mathbf{C}}$. Here $C_i^j(k)$ is the matrix defined by

$$\text{Coad}_k t_i = C_i^j(k) t_j. \quad (5.73)$$

The formula (5.72) can be inferred from

$$a = \exp(a^\mu t_\mu^\varepsilon) = \exp(a^\mu \varepsilon H^\mu) = 1 + a^\mu t_\mu^\varepsilon + O(\varepsilon^2) \quad (5.74)$$

and from the relation (4.5) rewritten as

$$[T^i, t_j^\varepsilon] = f^{kj}_i t_k^\varepsilon + ([t_i^\varepsilon, t_k^\varepsilon], T^j)_\varepsilon T^k. \quad (5.75)$$

Here f^{kj}_i are the structure constants of \mathcal{G}_0 defined by the relation $[T^i, T^j] = f^{ij}_l T^l$. Now recall the formula (3.38) for the Abelian moment maps $\langle M(k, a^\mu), T^j \rangle$ studied in Section 3.1.3. It can be rewritten as

$$\langle M(k, a^\mu), T^j \rangle = \langle \beta_L(k\phi), T^j \rangle = a^\mu C_\mu^j(k). \quad (5.76)$$

Note that here the multiplication $k\phi$ is in the sense of T^*G_0 . We conclude that the ε -expansion (5.72) can be rewritten as

$$b(ka) = Dres_k(a) = 1 + \varepsilon \langle M(k, a^\mu), T^j \rangle t_j^1 + O(\varepsilon^2). \quad (5.77)$$

Inserting this expansion into the formulae (5.68), (5.69) and (5.70) and using the explicit form (5.38) of the dual basis t_j^1 is the chosen representation, we obtain (in the lowest nontrivial order $\propto \varepsilon^2$) the following formula

$$\{\langle M, T^i \rangle, \langle M, T^j \rangle\} = f^{ij}_l \langle M, T^l \rangle. \quad (5.78)$$

The result (5.78) coincides with the standard formula (3.41) of Section 3.1.3.

5.1.4 The classical solution

It is very easy to solve classically the quasitriangular chiral geodesical model. It is enough to use the Hamiltonian (5.22) and the Poisson bracket (5.61) to conclude that the quasitriangular geodesics satisfy the equation

$$\frac{d}{d\tau} a_L^\mu = \{a_L^\mu, H_L\}_{q_{M_L}} = 0, \quad \frac{d}{d\tau} k_L = \{k_L, H_L\}_{q_{M_L}} = k_L T^\mu a_L^\mu. \quad (5.79)$$

The general solution of these equations has the form

$$k_L(\tau) = k_L(0) \exp(a_L^\mu T^\mu \tau), \quad a_L^\mu(\tau) = a_L^\mu(0). \quad (5.80)$$

By comparing with (3.18), we see that the deformation has not changed the classical solutions of the geodesical model! So what got deformed after all? It is in fact the symplectic structure on the space of solutions (phase space) that got deformed. This means that the natural dynamical variables of the group theoretical origin will have modified Poisson bracket and, upon the quantization, modified commutation relations. For instance, the correlation functions (in the field theoretical applications) will change.

5.2 Quasitriangular chiral WZW model

We proceed in full analogy with the previous Section 5.1, where the chiral model was constructed from the full geodesical model. Thus the deformed

chiral master model is going to live on the same affine model space \tilde{M}_L as the non-deformed one (3.60). Its symplectic structure $\tilde{\omega}_L^q$ will be obtained by embedding the affine model space \tilde{M}_L (defined in Section 3.2.2) into the double $\tilde{\tilde{D}}$ and by pulling back the Semenov-Tian-Shansky form from $\tilde{\tilde{D}}$ to \tilde{M}_L by the embedding map. The chiral Hamiltonian on \tilde{M}_L will be the same as in the standard non-deformed master model (3.58). Then the two step symplectic reduction of this quasitriangular chiral master model will be the deformed chiral WZW theory.

5.2.1 Quasitriangular chiral master model

Consider the affine Lu-Weinstein-Soibelman double $\tilde{\tilde{D}}$ introduced in section 4.4. There is the distinguished subgroup $\tilde{A} \subset \tilde{\tilde{D}}$ that we shall call the Cartan subgroup. It is defined as

$$\tilde{A} = \mathbf{R}_Q \times \nu(A) \times \mathbf{R}_l. \quad (5.81)$$

Here the first copy \mathbf{R}_Q corresponds to the automorphisms Q obtained by exponentiating the generator $\tilde{t}_\infty^1 = (1, 0, 0)$ and the second copy \mathbf{R}_l to the (central) line generated by $\tilde{t}_0^1 = (0, 0, 1)$.

Recall that ν is the injective homomorphism sending the group $B = L_+G_0^C$ into $\tilde{\tilde{D}}$ (cf. Section 4.4.2). Moreover, we have already encountered the group A in section 4.1.3. We have $A \subset B_0 \subset B$ hence the notation $\nu(A)$ makes sense. Note that the group $B_0 = AN$ is the zero mode subgroup of $B = L_+G_0^C$, hence the automorphisms from Q do not act on A . From this fact we conclude that the Cartan subgroup $\tilde{A} \subset \tilde{\tilde{D}}$ is commutative and the direct products in (5.81) make sense.

Recall that the Cartan subgroup $\tilde{A} \subset T^*\tilde{G}$ was defined as $\tilde{Y}^{-1}(\tilde{T})$, where $\tilde{T} = \mathcal{T} \oplus \mathbf{R}(i, 0, 0) \oplus \mathbf{R}(0, 0, i)$ and $\mathcal{T} = Lie(\mathbf{T})$. Consider the identification map $\tilde{\Lambda} : \tilde{\mathcal{G}}^* \rightarrow \tilde{\mathcal{B}} = Lie(\tilde{B})$ defined by

$$(\tilde{\Lambda}(\tilde{x}^*), \tilde{y})_{\tilde{\tilde{D}}} = \langle \tilde{x}^*, \tilde{y} \rangle, \quad \tilde{x}^* \in \tilde{\mathcal{G}}^*, \tilde{y} \in \tilde{\mathcal{G}}. \quad (5.82)$$

Note that $\tilde{\Lambda}$ depends on ε because $(\cdot, \cdot)_{\tilde{\tilde{D}}}$ does (cf. (4.147)). We can naturally define the fundamental Weyl alcove \tilde{A}_+ in the Cartan subgroup $\tilde{A} \subset \tilde{\tilde{D}}$ as

$$\tilde{A}_+ = \exp \tilde{\Lambda}(\tilde{\mathcal{A}}_+), \quad (5.83)$$

where $\tilde{\mathcal{A}}_+$ is the fundamental Weyl alcove in the Cartan subgroup $\tilde{A} \subset T^*\hat{G}$ as explained in Section 3.2.1.

Recall that the Semenov-Tian-Shansky form $\tilde{\omega}$ on the Heisenberg double \tilde{D} is given by the formula

$$\tilde{\omega} = \frac{1}{2}(\tilde{b}_L^* \lambda_{\tilde{B}} \frown \tilde{g}_R^* \rho_{\tilde{G}})_{\tilde{D}} + \frac{1}{2}(\tilde{b}_R^* \rho_{\tilde{B}} \frown \tilde{g}_L^* \lambda_{\tilde{G}})_{\tilde{D}}. \quad (5.84)$$

Here the maps $\tilde{b}_L : \tilde{D} \rightarrow \tilde{B}$ and $\tilde{g}_R : \tilde{D} \rightarrow \tilde{G}$ are induced by the decomposition $\tilde{D} = \tilde{B}\tilde{G}$ and $\tilde{g}_L : \tilde{D} \rightarrow \tilde{G}$ and $\tilde{b}_R : \tilde{D} \rightarrow \tilde{B}$ by $\tilde{D} = \tilde{G}\tilde{B}$. The expression $\lambda_{\tilde{G}}$ ($\rho_{\tilde{G}}$) denotes the left(right) invariant \tilde{G} -valued Maurer-Cartan form on the group \tilde{G} .

Consider now the affine model space $\tilde{M}_L = \tilde{G} \times \tilde{\mathcal{A}}_+$. The Semenov-Tian-Shansky form $\tilde{\omega}$ can be pulled back to the space \tilde{M}_L by the map $\tilde{\Xi} : \tilde{M}_L \rightarrow \tilde{D}$ defined by $\tilde{\Xi}(\tilde{k}, \tilde{a}) = \tilde{k}\tilde{a} \in \tilde{D}$, where $\tilde{k} \in \tilde{G}$, $\tilde{a} = \exp \tilde{\Lambda}(\tilde{\phi})$ and $\tilde{\phi} \in \tilde{\mathcal{A}}_+$. The pull-back form $\tilde{\omega}_L \equiv \tilde{\Xi}^* \tilde{\omega}$ defines the chiral symplectic structure on \tilde{M}_L . Its explicit form can be found by reconducting step by step the finite dimensional calculation of Section 5.1.1. The result is

$$\begin{aligned} \tilde{\omega}_L \equiv \tilde{\Xi}^* \tilde{\omega} = & \frac{1}{2}(d\tilde{b}_L(\tilde{k}\tilde{a})\tilde{b}_L^{-1}(\tilde{k}\tilde{a}) \frown d\tilde{p}\tilde{p}^{-1})_{\tilde{D}} + \\ & + (d\tilde{a}\tilde{a}^{-1} \frown \tilde{k}^{-1}d\tilde{k})_{\tilde{D}} + \frac{1}{2}(\tilde{k}^{-1}d\tilde{k} \frown \tilde{a}(\tilde{k}^{-1}d\tilde{k})\tilde{a}^{-1})_{\tilde{D}}, \end{aligned} \quad (5.85)$$

where $\tilde{p} = \tilde{k}\tilde{a}\tilde{k}^{-1}$. This leads to the following definition

Definition 5.6: The quasitriangular chiral master model is the dynamical system on the phase space \tilde{M}_L , whose symplectic structure is given by (5.85) and whose Hamiltonian is

$$\tilde{H}_L = -\frac{1}{2\kappa}(\tilde{\phi}, \tilde{\phi})_{\tilde{G}}. \quad (5.86)$$

Note that the Hamiltonian (5.86) is the same as that of the standard chiral master model (3.58), however, the symplectic structure is different. It is also crucial to remark that the model (5.85-86) is Poisson-Lie symmetric in the sense of Definition 4.7. The proof of this fact is very similar as that of the analogous finite-dimensional result expressed in Lemma 5.5. In particular, the non-Abelian moment map corresponding to the left \tilde{G} -action on \tilde{M}_L is given by $\tilde{b}_L(\tilde{k}\tilde{a})$.

5.2.2 Chiral symplectic reduction: the first step

Now we are going to perform the symplectic reduction of the chiral master model introduced in the previous subsection. As we know, we can do it in the language of the symplectic forms (like in Section 2.2.3) or using the Poisson bracket formalism (as in Section 7.3). Here we choose the first (second) possibility for the first (second) step of the reduction.

We first need some preliminary description of the objects with which we are going to work. The Cartan subgroup \hat{A} of the first floor double $\hat{\hat{D}}$ is defined as

$$\hat{A} = \mathbf{R}_Q \times \nu(A), \quad (5.87)$$

with the same notation as in (5.81). Then define the identification map $\hat{\Lambda} : \hat{\mathcal{G}}^* \rightarrow \hat{\mathcal{B}} = Lie(\hat{B})$ as follows

$$(\hat{\Lambda}(\hat{x}^*), \hat{y})_{\hat{D}} = \langle \hat{x}^*, \hat{y} \rangle, \quad \hat{x}^* \in \hat{\mathcal{G}}^*, \hat{y} \in \hat{\mathcal{G}}. \quad (5.88)$$

Recall also the definition of the alcove $\hat{\mathcal{A}}_+$:

$$\hat{\mathcal{A}}_+ = \{\hat{\phi} \in \tilde{\mathcal{A}}_+, \hat{\phi} = (0, \phi, a^\infty)^*\}.$$

The hat-alcove \hat{A}_+ in $\hat{\hat{D}}$ is then set to be

$$\hat{A}_+ = \exp \hat{\Lambda}(\hat{\mathcal{A}}_+). \quad (5.89)$$

Consider now the reduced model space $\hat{M}_L = \hat{G} \times \hat{\mathcal{A}}_+$. The Semenov-Tian-shansky form $\hat{\omega}$ on $\hat{\hat{D}}$ can be pulled back to the reduced model space \hat{M}_L by the map $\hat{\Xi} : \hat{M}_L \rightarrow \hat{\hat{D}}$ defined by $\hat{\Xi}(\hat{k}, \hat{a}) = \hat{k}\hat{a} \in \hat{\hat{D}}$, where $\hat{k} \in \hat{G}$, $\hat{a} = \exp \hat{\Lambda}(\hat{\phi})$ and $\hat{\phi} \in \hat{\mathcal{A}}_+$. The pull-back form $\hat{\omega}_L \equiv \hat{\Xi}^* \hat{\omega}$ defines the symplectic structure on \hat{M}_L :

$$\begin{aligned} \hat{\omega}_L \equiv \hat{\Xi}^* \hat{\omega} &= \frac{1}{2} (d\hat{b}_L(\hat{k}\hat{a})\hat{b}_L^{-1}(\hat{k}\hat{a}) \frown d\hat{p}\hat{p}^{-1})_{\hat{D}} + \\ &+ (d\hat{a}\hat{a}^{-1} \frown \hat{k}^{-1}d\hat{k})_{\hat{D}} + \frac{1}{2} (\hat{k}^{-1}d\hat{k} \frown \hat{a}(\hat{k}^{-1}d\hat{k})\hat{a}^{-1})_{\hat{D}}, \end{aligned} \quad (5.90)$$

where $\hat{p} = \hat{k}\hat{a}\hat{k}^{-1}$.

We note that due to the fact that $\tilde{G} = \mathbf{R} \times_S \hat{G}$, the affine model space \tilde{M}_L is diffeomorphic to $\hat{M}_L \times \mathbf{R}_S \times \mathbf{R}_l$. The natural additive coordinates

on \mathbf{R}_l and \mathbf{R}_s can be denoted as a^0 and s , respectively. Thus any element $(\tilde{k}, \tilde{\phi})$ of \tilde{M}_L can be naturally represented by the quadruple $(\hat{k}, \hat{\phi}, a^0, s)$, where $\tilde{k} = e^{s\tilde{T}^0} \hat{k}$, and $\tilde{a} = \hat{a} e^{a^0 \tilde{\Upsilon}(\tilde{t}_0)}$.

In order to perform the symplectic reduction, it is useful to change the coordinates on \tilde{M}_L . For this, recall that the maps $m^0 : \tilde{B} \rightarrow \mathbf{R}$ and $\hat{m} : \tilde{B} \rightarrow \hat{B}$ (cf. Conventions 4.12) were induced by the decomposition $\tilde{B} = \hat{B} \times \mathbf{R}_l$. We shall write occasionally $m(\tilde{b}) = \tilde{b}^0$ nad $\hat{m}(\tilde{b}) = \tilde{b}'$. We define also the maps $m_L^0 : \tilde{D} \rightarrow \mathbf{R}$ and $\hat{m}_L : \tilde{D} \rightarrow \hat{B}$ as $m_L^0 = m^0 \circ \tilde{b}_L$ and $\hat{m}_L = \hat{m} \circ \tilde{b}_L$. The essence of the first step of the symplectic reduction is contained in the following proposition:

Theorem 5.7: 1) It holds

$$m_L^0(\hat{k}, \hat{\phi}, a^0, s) = a^0 + (\widetilde{Dres_{\hat{k}} \hat{a}})^0, \quad \hat{a} = \exp \hat{\Lambda}(\hat{\phi}); \quad (5.91)$$

2) In the coordinates $(\hat{k}, \hat{\phi}, m_L^0, s)$ on \tilde{M}_L , the symplectic form $\tilde{\omega}_L$ can be written as

$$\tilde{\omega}_L = -ds \wedge dm_L^0 + \hat{\omega}_L, \quad (5.92)$$

where $\hat{\omega}_L$ is the symplectic form on \hat{M}_L defined in (5.90).

The proof of this theorem will necessitate the following lemma:

Lemma 5.8: It holds

$$\hat{m}_L(\widetilde{Ad_{\hat{k}} \hat{a}}) = \hat{b}_L(\widetilde{Ad_{\hat{k}} \hat{a}}), \quad \tilde{g}_R(\widetilde{Ad_{\hat{k}} \hat{a}}) = \hat{g}_R(\widetilde{Ad_{\hat{k}} \hat{a}}). \quad (5.93)$$

This lemma is proved in the Appendix 7.4

Proof of the Theorem 5.7: 1) We have

$$\tilde{b}_L(e^{s\tilde{T}^0} \hat{k} \hat{a} e^{a^0 \tilde{\Upsilon}(\tilde{t}_0)}) = e^{a^0 \tilde{\Upsilon}(\tilde{t}_0)} \tilde{b}_L(e^{s\tilde{T}^0} \hat{k} \hat{a}). \quad (5.94)$$

because the line $e^{a^0 \tilde{\Upsilon}(\tilde{t}_0)}$ is central in \tilde{D} . We stress that the multiplication $\hat{k} \hat{a}$ in (5.94) is taken in \tilde{D} . Then we use the fact that the automorphism S preserves both subgroups \tilde{G} and \tilde{B} . This permits to write

$$\tilde{b}_L(e^{s\tilde{T}^0} \hat{k} \hat{a}) =$$

$$\tilde{b}_L(e^{s\tilde{T}^0}\hat{k}\hat{a}e^{-s\tilde{T}^0}) = e^{s\tilde{T}^0}\tilde{b}_L(\hat{k}\hat{a})e^{-s\tilde{T}^0} = (\exp(m_L^0(\hat{k}\hat{a})\tilde{Y}(\tilde{t}_0)))e^{s\tilde{T}^0}\hat{m}_L(\hat{k}\hat{a})e^{-s\tilde{T}^0}, \quad (5.95)$$

where the maps m^0, \hat{m} were defined in Conventions 4.12. Since S preserve also \tilde{B} , it follows from (5.94) and (5.95) that

$$m_L^0(\hat{k}, \hat{\phi}, a^0, s) = a^0 + m_L^0(\hat{k}\hat{a}) \equiv a^0 + (\widetilde{Dres_k}\hat{a})^0; \quad (5.96)$$

2) We have needed the statement 1) in order to show that the coordinate a^0 can be traded for the new coordinate m_L^0 . So from now until the end of the proof we shall work with the variables $(\hat{k}, \hat{\phi}, m_L^0, s)$ on \tilde{M}_L . The quasitriangular chiral master model is Poisson-Lie symmetric, hence we know that the group \tilde{G} acts on the affine model space \tilde{M}_L in the Poisson-Lie way. Recall that this follows from the fact that \tilde{M}_L can be viewed as the submanifold of \tilde{D} preserved by the \tilde{G} -action on \tilde{D} . The latter is of the Poisson-Lie type by construction and it has the non-Abelian moment map given by the factorization map $\tilde{b}_L : \tilde{D} \rightarrow \tilde{B}$. The restriction of the map \tilde{b}_L to the affine model space \tilde{M}_L is the Poisson-Lie moment map on \tilde{M}_L .

The relation (4.87) proved in Section 5.1.2 implies the following formula

$$\tilde{\omega}_L(., v_{\tilde{T}^0}) = (\tilde{T}^0, d\tilde{b}_L\tilde{b}_L^{-1})_{\tilde{B}}. \quad (5.97)$$

Here on the l.h.s., $v_{\tilde{T}^0} = \frac{\partial}{\partial s}$ is viewed as the vector field on \tilde{M}_L corresponding to the left action of the generator \tilde{T}^0 and on the r.h.s., \tilde{T}^0 is viewed simply as the element of $\tilde{\mathcal{G}}$.

Now recall the definition of the map $m^0 : \tilde{B} \rightarrow \mathbf{R}$. It is induced by the decomposition $\tilde{B} = \hat{B} \times \mathbf{R}$. Since \mathbf{R} commutes with \hat{B} , it follows that

$$\tilde{\omega}_L(., v_{\tilde{T}^0}) = (\tilde{T}^0, d\tilde{b}_L\tilde{b}_L^{-1})_{\tilde{B}} = (\tilde{T}^0, \tilde{\Lambda}(\tilde{t}_0))_{\tilde{B}} dm_L^0 = dm_L^0. \quad (5.98)$$

In other words, the structure of our WZW double \tilde{D} insures, that the \tilde{T}^0 -action on \tilde{M}_L is Hamiltonian in the standard (Abelian) sense.

It follows immediately from the formula (5.92), that

$$\tilde{\omega}_L = -ds \wedge dm_L^0 + \tilde{\Omega}(\hat{k}, \hat{\phi}, m_L^0) \quad (5.99)$$

where $\tilde{\Omega}$ does not contain ds . Moreover, $\tilde{\Omega}$ neither depends on s , because $\tilde{\omega}_L$ is invariant with respect to the Hamiltonian vector field $\partial/\partial s$. In fact, it is

evident that $\tilde{\Omega}$ is nothing but the pull-back of $\tilde{\omega}_L$ on the surface $s = 0$ in \tilde{M}_L . We can therefore write $\tilde{\Omega} = \tilde{\omega}_L(s = 0)$. Now we have from (5.85)

$$\begin{aligned} \tilde{\omega}_L(s = 0) &= \frac{1}{2}(d\tilde{b}_L(\hat{k}\tilde{a})\tilde{b}_L^{-1}(\hat{k}\tilde{a}) \frown d\tilde{p}_0\tilde{p}_0^{-1})_{\tilde{D}} + \\ &+ (d\tilde{a}\tilde{a}^{-1} \frown \hat{k}^{-1}d\hat{k})_{\tilde{D}} + \frac{1}{2}(\hat{k}^{-1}d\hat{k} \frown \tilde{a}(\hat{k}^{-1}d\hat{k})\tilde{a}^{-1})_{\tilde{D}}, \end{aligned} \quad (5.100)$$

where $\tilde{p}_0 = \widetilde{Ad}_{\hat{k}}\tilde{a}$ and $\tilde{a} = \hat{a} \exp(a^0(m_L^0, \hat{k}, \hat{a})\tilde{\Upsilon}(\tilde{t}_0))$. The reader should pay attention to the distribution of the hats and tildes and to the fact that the group multiplication in this formula is taken in \tilde{D} . Now we study (5.100) term by term:

$$(d\tilde{a}\tilde{a}^{-1} \frown \hat{k}^{-1}d\hat{k})_{\tilde{D}} = (d\hat{a}\hat{a}^{-1} \frown \hat{k}^{-1}d\hat{k})_{\hat{D}}; \quad (5.101)$$

this follows from the fact that $(\hat{k}^{-1}d\hat{k}, \tilde{\Upsilon}(\tilde{t}_0))_{\tilde{D}} = 0$. Then we have

$$(\hat{k}^{-1}d\hat{k} \frown \tilde{a}(\hat{k}^{-1}d\hat{k})\tilde{a}^{-1})_{\tilde{D}} = (\hat{k}^{-1}d\hat{k} \frown \widetilde{Ad}_{\hat{a}}(\hat{k}^{-1}d\hat{k}))_{\tilde{D}}, \quad (5.102)$$

because $\exp(a^0\tilde{\Upsilon}(\tilde{t}_0))$ is in the center of \tilde{D} . Moreover, one checks directly that

$$(\hat{k}^{-1}d\hat{k} \frown \widetilde{Ad}_{\hat{a}}(\hat{k}^{-1}d\hat{k}))_{\tilde{D}} = (\hat{k}^{-1}d\hat{k} \frown \widetilde{Ad}_{\hat{a}}(\hat{k}^{-1}d\hat{k}))_{\hat{D}}. \quad (5.103)$$

It is first convenient to rewrite the remaining term as

$$\begin{aligned} &\frac{1}{2}(d\tilde{b}_L(\hat{k}\tilde{a})\tilde{b}_L^{-1}(\hat{k}\tilde{a}) \frown d\tilde{p}_0\tilde{p}_0^{-1})_{\tilde{D}} = \\ &= \frac{1}{2}(\tilde{b}_L^{-1}(\widetilde{Ad}_{\hat{k}}\tilde{a})d\tilde{b}_L(\widetilde{Ad}_{\hat{k}}\tilde{a}) \frown d\tilde{g}_R(\widetilde{Ad}_{\hat{k}}\tilde{a})\tilde{g}_R^{-1}(\widetilde{Ad}_{\hat{k}}\tilde{a}))_{\tilde{D}}. \end{aligned} \quad (5.104)$$

We know from Lemma 5.8 that $\tilde{g}_R(\widetilde{Ad}_{\hat{k}}\tilde{a}) \in \hat{G}$, hence it follows

$$\begin{aligned} &\frac{1}{2}(d\tilde{b}_L(\hat{k}\tilde{a})\tilde{b}_L^{-1}(\hat{k}\tilde{a}) \frown d\tilde{p}_0\tilde{p}_0^{-1})_{\tilde{D}} = \\ &= \frac{1}{2}(\hat{m}_L^{-1}(\widetilde{Ad}_{\hat{k}}\hat{a})d\hat{m}_L(\widetilde{Ad}_{\hat{k}}\hat{a}) \frown d\tilde{g}_R(\widetilde{Ad}_{\hat{k}}\hat{a})\tilde{g}_R^{-1}(\widetilde{Ad}_{\hat{k}}\hat{a}))_{\tilde{D}}, \end{aligned} \quad (5.105)$$

where $\hat{m}_L = \hat{m} \circ \tilde{b}_L$. We use again the Lemma 5.8 to rewrite finally (5.105) into

$$\frac{1}{2}(d\tilde{b}_L(\hat{k}\tilde{a})\tilde{b}_L^{-1}(\hat{k}\tilde{a}) \frown d\tilde{p}_0\tilde{p}_0^{-1})_{\tilde{D}} =$$

$$= \frac{1}{2}(\hat{b}_L^{-1}(\hat{A}d_{\hat{k}}\hat{a})d\hat{b}_L(\hat{A}d_{\hat{k}}\hat{a}) \frown d\hat{g}_R(\hat{A}d_{\hat{k}}\hat{a})\hat{g}_R^{-1}(\hat{A}d_{\hat{k}}\hat{a}))_{\hat{D}}. \quad (5.106)$$

Collecting (5.101), (5.103) and (5.106), we conclude

$$\tilde{\omega}_L(s=0) = \hat{\omega}_L$$

The theorem is proved. #

The first step of the symplectic reduction now consists in setting $m_L^0 = 0$. This is consistent, since the Hamiltonian $\tilde{H}_L = -\frac{1}{2\kappa}(\tilde{\phi}, \tilde{\phi})_{\tilde{G}^*}$ Poisson commutes with m_L^0 because it is obviously invariant with respect to the left \tilde{G} -action on \tilde{M}_L .

From the Theorem 5.7 one concludes immediately that the reduced dynamical system has \hat{M}_L for its phase space, its Hamiltonian reads

$$\hat{H}_L = -\frac{1}{2\kappa}(\phi, \phi)_{\mathcal{G}^*} - \frac{a^\infty}{\kappa}(\widetilde{Dres_{\hat{k}}\hat{a}})^0, \quad (5.107)$$

where $\hat{a} = e^{\hat{\Lambda}(\hat{\phi})}$, $\hat{\phi} = a^\infty \hat{t}_\infty + \pi^*(\phi)$ and its symplectic form is

$$\begin{aligned} \hat{\omega}_L = & \frac{1}{2}(d\hat{b}_L(\hat{k}\hat{a})\hat{b}_L^{-1}(\hat{k}\hat{a}) \frown d\hat{p}\hat{p}^{-1})_{\hat{D}} + \\ & + (d\hat{a}\hat{a}^{-1} \frown \hat{k}^{-1}d\hat{k})_{\hat{D}} + \frac{1}{2}(\hat{k}^{-1}d\hat{k} \frown \hat{a}(\hat{k}^{-1}d\hat{k})\hat{a}^{-1})_{\hat{D}}, \end{aligned} \quad (5.108)$$

where $\hat{p} = \hat{k}\hat{a}\hat{k}^{-1}$.

5.2.3 Chiral symplectic reduction: the second step

In order to perform the second step of the reduction, we have to find the standard Hamiltonian charge generating the central circle action on \hat{M}_L . Its existence is guaranteed because we know that the affine Lu-Weinstein-Soibelman double $\tilde{\tilde{D}}$ is of the WZW type (cf. Definition 4.10). We find this charge from the fundamental relation (4.87)

$$\hat{\omega}_L(., v_{\hat{T}^\infty}) = (\hat{T}^\infty, d\hat{b}_L\hat{b}_L^{-1})_{\hat{D}} = dm_L^\infty(\hat{k}\hat{a}), \quad (5.109)$$

which holds because $\hat{\omega}_L$ was pulled back from the Drinfeld double $\hat{\tilde{D}}$. Here $v_{\hat{T}^\infty}$ is the vector field on \hat{M}_L corresponding to the left action of \hat{T}^∞ and

$m_L^\infty = m^\infty \circ \hat{b}_L$. Recall that $m^\infty : \hat{B} \rightarrow \mathbf{R}$ was defined by the decomposition $\hat{B} = \mathbf{R} \times_Q B$. It is easy to evaluate m_L^∞ , in fact, it holds $m_L^\infty(\hat{k}\hat{a}) = a^\infty$.

Fix now the submanifold of \hat{M}_L defined by $a^\infty = \kappa$ and consider the space of the central circle orbits on this submanifold. This space of orbits is nothing but the WZW model space $M_L^{WZ} = G \times \mathcal{A}_+^1$, where \mathcal{A}_+^1 is the standard Weyl alcove. Before giving the explicit description of the (doubly) reduced symplectic structure on M_L^{WZ} , we first write the reduced Hamiltonian H_L^{qWZ} on M_L^{WZ} . The consistency requires that the first floor Hamiltonian $\hat{H}_L(\hat{k}\hat{a})$ be invariant with respect to the central circle action. But this is evident since

$$\widetilde{Dres}_{\hat{k}}\hat{a} = \tilde{b}(\hat{k}\hat{a}) = \tilde{b}(\widetilde{Ad}_{\hat{k}}\hat{a}). \quad (5.110)$$

Thus we obtain

$$H_L^{qWZ}(k, a^\mu) = -\frac{1}{2\kappa}(\phi, \phi)_{\mathcal{G}^*} - (\widetilde{Dres}_{\hat{k}}e^{\hat{\Lambda}(\hat{\phi}_\kappa)})^0, \quad (5.111)$$

where $\hat{\phi}_\kappa = \kappa\hat{t}_\infty + \pi^*(\phi)$, $\phi = \kappa a^\mu t_\mu$ and $a^\mu t_\mu \in \mathcal{A}_+^1$. The Hamiltonian H_L^{qWZ} depends on ε due to the ε -dependence of $\hat{\Lambda}$ and of the map $m^0 : \hat{B} \rightarrow \mathbf{R}$. We shall see in Section 5.2.7 that in the $\varepsilon \rightarrow 0$ (or $q \rightarrow 1$) limit the formula (5.111) yields the standard chiral Sugawara Hamiltonian H_L^{WZ} given by (3.118).

Let us now study the quasitriangular symplectic structure on M_L^{WZ} . As we have already said, it is convenient to perform the second step of the symplectic reduction by working with the Poisson brackets. This means that we shall first need to invert the symplectic form $\hat{\omega}_L$ to obtain the corresponding Poisson bivector $\hat{\Sigma}_L$ on \hat{M}_L . Our strategy for inverting $\hat{\omega}_L$ will rely on using its Poisson-Lie symmetry. Indeed, the story of the finite-dimensional Section 5.1.2 can be directly used also in our affine situation if we replace G_0 and \mathcal{A}_+^0 of section 5.1.2 by our \hat{G} and $\hat{\mathcal{A}}_+$, respectively. Thus we have

$$\hat{\omega}_L(., v_{\hat{T}}) = (\hat{T}, d\hat{b}_L(\hat{k}\hat{a})\hat{b}_L^{-1}(\hat{k}\hat{a}))_{\hat{B}}, \quad (5.112)$$

where \hat{T} is any generator of $\hat{\mathcal{G}}$ and $v_{\hat{T}}$ is the vector field on \hat{M}_L corresponding to the left action of \hat{T} on \hat{M}_L . Following the reasoning after the proof of Lemma 5.5, we conclude that the Poisson bivector $\hat{\Sigma}_L$ on \hat{M}_L fulfils

$$\mathcal{L}_{\hat{v}}\hat{\Sigma}_L = -\hat{v} \wedge \hat{v}, \quad (5.113)$$

where $\hat{v} \in T\hat{M}_L \otimes \hat{\mathcal{G}}^* = T\hat{M}_L \otimes \mathcal{B}$ generates the left action of \hat{G} on \hat{M}_L and $\mathcal{L}_{\hat{v}}$ is the Lie derivative. From (5.113), it follows (cf. (5.37)) that $\hat{\Sigma}_L$ must be of the form

$$\begin{aligned} \hat{\Sigma}_L(\hat{k}, \hat{\phi}) = & -\frac{1}{2}(\Pi_{\hat{G}}^R)_{ij}(\hat{k})R_{\hat{k}*}\hat{T}^i \wedge R_{\hat{k}*}\hat{T}^j + \\ & + \hat{\Sigma}_{ij}^0(\hat{\phi})L_{\hat{k}*}\hat{T}^i \wedge L_{\hat{k}*}\hat{T}^j + \hat{\sigma}_i^{\hat{\mu}}(\hat{\phi})L_{\hat{k}*}\hat{T}^i \wedge \frac{\partial}{\partial \phi^{\hat{\mu}}} + \hat{s}^{\hat{\mu}\hat{\nu}}(\hat{\phi})\frac{\partial}{\partial \phi^{\hat{\mu}}} \wedge \frac{\partial}{\partial \phi^{\hat{\nu}}}. \end{aligned} \quad (5.114)$$

Here $\hat{\phi} = \phi^\infty \hat{t}_\infty + \phi^\mu \pi^*(t_\mu) \equiv a^\infty \hat{t}_\infty + a^\infty a^\mu \pi^*(t_\mu)$, and the notation $\phi^{\hat{\mu}}$ means that we consider at the same time the coordinates ϕ^μ and ϕ^∞ ; in other words, the subscript $\hat{\mu}$ runs over μ and over ∞ . The expression $\frac{1}{2}(\Pi_{\hat{G}}^R)_{ij}(\hat{k})R_{\hat{k}*}\hat{T}^i \wedge R_{\hat{k}*}\hat{T}^j$ is the Poisson-Lie bivector on the group \hat{G} written in some basis \hat{T}^i of $\hat{\mathcal{G}}$. Recall that the Poisson-Lie bracket on \hat{G} is entirely specified by the structure of the affine Lu-Weinstein-Soibelman double $\hat{\hat{D}}$, as it was explained in section 4.1.2.

Thus we again observe the power of the Poisson-Lie symmetry: the Poisson tensor $\hat{\Sigma}_L$ on the affine model space \hat{M}_L is completely determined by its value at the points $(\hat{e}, \hat{\phi}) \in \hat{M}_L$, in other words: at the unit element \hat{e} of \hat{G} .

Our next task is to find the coefficient functions $\hat{\Sigma}_{ij}^0(\hat{\phi})$, $\hat{\sigma}_i^{\hat{\mu}}(\hat{\phi})$ and $\hat{s}^{\hat{\mu}\hat{\nu}}(\hat{\phi})$. For this, we first calculate the matrix of the symplectic form $\hat{\omega}_L$ at points $(\hat{e}, \hat{\phi})$. We do it in the basis $R_{\hat{a}*}\hat{T}^i, R_{\hat{a}*}\hat{\Lambda}(\hat{t}_{\hat{\mu}})$ of the $\hat{\Xi}$ -pushed-forward tangent space $T_{(\hat{e}, \hat{\phi})}\hat{M}_L$; recall that $\hat{a} = \exp \hat{\Lambda}(\hat{\phi})$ and we set $\hat{t}_{\hat{\mu}} = (\hat{t}_\infty, \pi^*(t_\mu))$. We then invert this matrix to obtain the unknown coefficient functions of the Poisson bivector.

The convenient basis of $\hat{\mathcal{G}} = \widehat{L\mathcal{G}_0}$ reads

$$\hat{T}^i = \hat{T}^\infty, \iota(T^\mu), \iota(B^{\hat{\alpha}}), \iota(C^{\hat{\alpha}}), \quad \hat{\alpha} \in \hat{\Phi}_+. \quad (5.115)$$

Recall that

$$T^\mu = iH^\mu, \quad B^{\hat{\alpha}} = \frac{i}{\sqrt{2}}(E^{\hat{\alpha}} + E^{-\hat{\alpha}}), \quad C^{\hat{\alpha}} = \frac{1}{\sqrt{2}}(E^{\hat{\alpha}} - E^{-\hat{\alpha}}), \quad (5.116)$$

where

$$E^{\hat{\alpha}} = E^\alpha e^{in\sigma}, \quad \hat{\alpha} = (\alpha, n); \quad E^{\hat{\alpha}} = H^\mu e^{in\sigma}, \quad \hat{\alpha} = (\mu, n \neq 0). \quad (5.117)$$

The complete list of the conventions and notations concerning this basis can be found in Section 3.2.4. The corresponding dual basis of $\hat{\mathcal{G}}^*$ is

$$\hat{t}_i = \hat{t}_\infty, \pi^*(t_\mu), \pi^*(b_{\hat{\alpha}}), \pi^*(c_{\hat{\alpha}}), \quad \hat{\alpha} \in \hat{\Phi}_+. \quad (5.118)$$

From this we obtain the dual basis of $\hat{\mathcal{B}}$; it reads

$$\hat{\Lambda}(\hat{t}_i) = \varepsilon \hat{t}_\infty^1, \varepsilon \nu_*(H^\mu), \varepsilon \nu_*\left(\frac{|\hat{\alpha}|^2}{\sqrt{2}} E^{\hat{\alpha}}\right), \varepsilon \nu_*\left(-i \frac{|\hat{\alpha}|^2}{\sqrt{2}} E^{\hat{\alpha}}\right), \quad \hat{\alpha} \in \hat{\Phi}_+, \quad (5.119)$$

where $\hat{t}_\infty^1 = -i\partial_\sigma$ and $\nu_* : \mathcal{B} \rightarrow \hat{\mathcal{B}}$ were defined in Section 4.4.2. Moreover, $|\hat{\alpha}|^2 = |\alpha|^2$ for $\hat{\alpha} = (\alpha, n)$ and $|\hat{\alpha}|^2 = 2$ for $\hat{\alpha} = (\mu, n)$. Of course, $\hat{\mathcal{B}}$ is the Lie algebra of $\hat{B} \subset \hat{\mathcal{D}}$ and $\mathcal{B} = \text{Lie}(B)$ where $L_+ G_0^{\mathcal{C}} = B \subset D = L G_0^{\mathcal{C}}$. The dual basis depends on ε and satisfies the basic condition $(\hat{\Lambda}(\hat{t}_i), \hat{T}^j)_{\hat{\mathcal{D}}} = \delta_i^j$.

Recall that $\hat{a} = \exp \hat{\Lambda}(\hat{\phi})$ is the element of the alcove $\hat{A}_+ \subset \hat{A}$; the Lie algebra \hat{A} is generated by the generators $\hat{\Lambda}(\hat{t}_{\hat{\mu}}) = \hat{\Lambda}(\hat{t}_\infty, \pi^*(t_\mu)) = (\varepsilon \nu_*(H^\mu), -i\varepsilon \partial_\sigma)$. We decompose the elements of the basis of $T_{\hat{a}} \hat{\Xi}(\hat{M}_L)$ into two parts: the $\hat{\alpha}$ -part generated by $R_{\hat{a}*} \iota(B^{\hat{\alpha}}), R_{\hat{a}*} \iota(C^{\hat{\alpha}})$, $\hat{\alpha} \in \hat{\Phi}_+$ and the $\hat{\mu}$ -part generated by $R_{\hat{a}*} \hat{T}^{\hat{\mu}}, R_{\hat{a}*} \hat{\Lambda}(\hat{t}_{\hat{\mu}})$. Here by $\hat{T}^{\hat{\mu}}$ we mean either $\iota(T^\mu)$ or \hat{T}^∞ . Now we use the formula (4.18) from Section 4.1.2, expressing the Semenov-Tian-Shansky form on \hat{D} .

$$\hat{\omega}(t, u) = (t, (\Pi_{\tilde{L}R} - \Pi_{L\tilde{R}})u)_{\hat{\mathcal{D}}}.$$

We immediately observe that

$$\hat{\omega}_L(\hat{\alpha}, \hat{\mu}) = 0, \quad \hat{\omega}_L(\hat{\alpha}, \hat{\beta}) = 0, \quad \hat{\alpha}, \hat{\beta} \in \hat{\Phi}_+, \quad \hat{\alpha} \neq \hat{\beta}. \quad (5.120)$$

This is because we are at the point $(\hat{e}, \hat{a}) \in \hat{M}_L(\subset \hat{D})$. It is hence sufficient to invert the matrix $\hat{\omega}_L$ for the $\hat{\mu}$ -sector and for every $\hat{\alpha}$ -sector separately. We have

$$\Pi_{L\tilde{R}} R_{\hat{a}*} \hat{T}^{\hat{\mu}} = 0, \quad \Pi_{\tilde{L}R} R_{\hat{a}*} \hat{T}^{\hat{\mu}} = R_{\hat{a}*} \hat{T}^{\hat{\mu}}. \quad (5.121)$$

From this we obtain

$$\omega_L(R_{\hat{a}*} \hat{\Lambda}(\hat{t}_{\hat{\nu}}), R_{\hat{a}*} \hat{T}^{\hat{\mu}}) = \delta_{\hat{\nu}}^{\hat{\mu}}. \quad (5.122)$$

One has

$$R_{\hat{a}*} \iota(C^{\hat{\alpha}}) = (R_{\hat{a}*} \iota(C^{\hat{\alpha}}), L_{\hat{a}*} \hat{\Lambda}(\hat{t}_i))_{\hat{\mathcal{D}}} L_{\hat{a}*} \hat{T}^i + (R_{\hat{a}*} \iota(C^{\hat{\alpha}}), L_{\hat{a}*} \hat{T}^i)_{\hat{\mathcal{D}}} L_{\hat{a}*} \hat{\Lambda}(\hat{t}_i) \quad (5.123)$$

and

$$L_{\hat{a}*}\hat{\Lambda}(b_{\hat{\alpha}}) = e^{\frac{\varepsilon}{2}|\hat{a}|^2\langle i\hat{\alpha}^\vee, \hat{\phi} \rangle} R_{\hat{a}*}\hat{\Lambda}(b_{\hat{\alpha}}), \quad L_{\hat{a}*}\hat{\Lambda}(c_{\hat{\alpha}}) = e^{\frac{\varepsilon}{2}|\hat{a}|^2\langle i\hat{\alpha}^\vee, \hat{\phi} \rangle} R_{\hat{a}*}\hat{\Lambda}(c_{\hat{\alpha}}). \quad (5.124)$$

Then we have for every $\hat{\alpha} \in \hat{\Phi}_+$

$$\begin{aligned} \Pi_{L\hat{R}} R_{\hat{a}*}\iota(C^{\hat{\alpha}}) = \\ e^{\frac{\varepsilon}{2}|\hat{a}|^2\langle i\hat{\alpha}^\vee, \hat{\phi} \rangle} \left[(L_{\hat{a}*}\iota(B^{\hat{\alpha}}), R_{\hat{a}*}\iota(C^{\hat{\alpha}}))_{\hat{D}} R_{\hat{a}*}\hat{\Lambda}(b_{\hat{\alpha}}) + (L_{\hat{a}*}\iota(C^{\hat{\alpha}}), R_{\hat{a}*}\iota(C^{\hat{\alpha}}))_{\hat{D}} R_{\hat{a}*}\hat{\Lambda}(c_{\hat{\alpha}}) \right]; \end{aligned} \quad (5.125)$$

$$\Pi_{\bar{L}R} R_{\hat{a}*}\iota(C^{\hat{\alpha}}) = R_{\hat{a}*}\iota(C^{\hat{\alpha}}). \quad (5.126)$$

It follows that

$$\begin{aligned} \hat{\omega}_L(R_{\hat{a}*}\iota(B^{\hat{\alpha}}), R_{\hat{a}*}\iota(C^{\hat{\alpha}})) = \\ = -e^{\frac{\varepsilon}{2}|\hat{a}|^2\langle i\hat{\alpha}^\vee, \hat{\phi} \rangle} (L_{\hat{a}*}\iota(B^{\hat{\alpha}}), R_{\hat{a}*}\iota(C^{\hat{\alpha}}))_{\hat{D}} = \frac{1}{\varepsilon|\hat{\alpha}|^2} (e^{2\frac{\varepsilon}{2}|\hat{a}|^2\langle i\hat{\alpha}^\vee, \hat{\phi} \rangle} - 1). \end{aligned} \quad (5.127)$$

Recall from Section 3.2.4, that for $\hat{\alpha} = (\alpha, n)$:

$$\frac{\varepsilon}{2}|\hat{a}|^2\langle i\hat{\alpha}^\vee, \hat{\phi} \rangle = \varepsilon a^\infty (a^\mu \langle \alpha, H^\mu \rangle + n) \quad (5.128)$$

and for $\hat{\alpha} = (\mu, n)$

$$\frac{\varepsilon}{2}|\hat{a}|^2\langle i\hat{\alpha}^\vee, \hat{\phi} \rangle = \varepsilon a^\infty n. \quad (5.129)$$

Our conclusion is that

$$\hat{s}^{\hat{\mu}\hat{\nu}} = 0, \quad \hat{\sigma}_{\hat{\alpha}}^{\hat{\mu}} = 0, \quad \hat{\sigma}_{\hat{\mu}}^{\hat{\nu}} = \delta_{\hat{\mu}}^{\hat{\nu}}, \quad \hat{\Sigma}_{\hat{\mu}\hat{\nu}}^0 = \hat{\Sigma}_{\hat{\alpha}\hat{\mu}}^0 = \hat{\Sigma}_{\hat{\alpha}\hat{\beta}}^0 = 0 \quad (5.130)$$

and the explicit formula for the Poisson bivector $\hat{\Sigma}_L$ on \hat{M}_L reads

$$\begin{aligned} \hat{\Sigma}_L(\hat{k}, \hat{\phi}^{\hat{\mu}}) = -\frac{1}{2}(\Pi_{\hat{G}}^R)_{ij}(\hat{k}) R_{\hat{k}*} \hat{T}^i \wedge R_{\hat{k}*} \hat{T}^j + \\ + \sum_{\hat{\alpha} \in \hat{\Phi}_+} \frac{\varepsilon|\hat{\alpha}|^2}{(1 - e^{2\frac{\varepsilon}{2}|\hat{a}|^2\langle i\hat{\alpha}^\vee, \hat{\phi} \rangle})} L_{\hat{k}*}\iota(B^{\hat{\alpha}}) \wedge L_{\hat{k}*}\iota(C^{\hat{\alpha}}) + L_{\hat{k}*} \hat{T}^{\hat{\mu}} \wedge \frac{\partial}{\partial \hat{\phi}^{\hat{\mu}}}. \end{aligned} \quad (5.131)$$

So far we have inverted the form $\hat{\omega}_L$, now we are ready to perform the second step of the symplectic reduction induced by setting $a^\infty = \kappa$. Consider a pair of functions ϕ_i , $i = 1, 2$ on M_L^{WZ} . We wish to calculate their reduced Poisson

bracket $\{\phi_1, \phi_2\}_{qWZ}$. The general procedure of the symplectic reduction at the level of the Poisson brackets is described in the Appendix 7.3. In our particular situation, it works as follows: define two functions $\hat{\phi}_i$ on \hat{M}_L as

$$\hat{\phi}_i(\hat{k}, a^\mu, a^\infty = \kappa) \equiv \phi_i(\pi(\hat{k}), a^\mu), \quad \hat{k} \in \widehat{LG}_0. \quad (5.132)$$

Calculate then the quasitriangular Poisson bracket $\{\hat{\phi}_1, \hat{\phi}_2\}_{q\hat{M}_L}$ on \hat{M}_L . It verifies

$$\{a^\infty, \{\hat{\phi}_1, \hat{\phi}_2\}_{q\hat{M}_L}\}_{q\hat{M}_L} = 0 \quad (5.133)$$

as the simple consequence of the Jacobi identity and the central circle invariance of $\hat{\phi}_i$. This means that it exists a function on M_L^{WZ} denoted suggestively as $\{\phi_1, \phi_2\}_{qWZ}$ which verifies

$$\{\hat{\phi}_1, \hat{\phi}_2\}_{q\hat{M}_L}(\hat{k}, a^\mu, a^\infty = \kappa) = \{\phi_1, \phi_2\}_{qWZ}(\pi(\hat{k}), a^\mu). \quad (5.134)$$

Needless to say, the function $\{\phi_1, \phi_2\}_{qWZ}$ is the seeked reduced Poisson bracket. This method we now apply for the functions ϕ_i of particular form.

5.2.4 Deformed affine dynamical r -matrix

Our next task will consist in computing the reduced Poisson bracket $\{k \otimes k\}_{qWZ}$. The reader who still did not get used to the notation for the matrix Poisson bracket should again consult Sections 3.1.3 and 3.2.6. We have

$$\{k \otimes k\}_{qWZ} = \{k \otimes k\}_0 - \{k \otimes k\}_G, \quad (5.135)$$

where the first bracket correspond (modulo the reduction) to the bivector on the second line on the r.h.s. of (5.131), and the second bracket to that on the first line. Of course, the relation (5.135) is the affine analogue of (5.49). Also the way of the evaluating of the two terms in (5.135) follows in spirit the calculation of Section 5.1.3. For doing it, we first introduce some necessary notions:

Define the identification map $\Lambda : \mathcal{G}^* \rightarrow \mathcal{B} = Lie(B)$ as usual

$$(\Lambda(x^*), y)_{\mathcal{D}} = \langle x^*, y \rangle, \quad x^* \in \mathcal{G}^*, y \in \mathcal{G}. \quad (5.136)$$

We can directly check from the definition (4.127), (4.128) of $(\cdot, \cdot)_{\hat{D}}$ that

$$\hat{\Lambda} \circ \pi^* = \nu_* \circ \Lambda. \quad (5.137)$$

Now using the relations (4.63) and (4.66), we first calculate the relevant Poisson-Lie bracket on $\hat{G} = \widehat{LG}_0$:

$$\begin{aligned} \{\pi(\hat{k}) \otimes \pi(\hat{k})\}_G^R &= - \left[\pi(\iota(T^i)\hat{k}) \otimes \pi(\iota(T^j)\hat{k}) \right] \times \\ &\times \left[(\hat{k}^{-1}(\hat{\Lambda} \circ \pi^*)(t_i)\hat{k}, (\hat{\Lambda} \circ \pi^*)(t_l))_{\hat{\mathcal{D}}} (\hat{k}^{-1}(\hat{\Lambda} \circ \pi^*)(t_j)\hat{k}, \iota(T^l))_{\hat{\mathcal{D}}} + \right. \\ &\quad \left. (\hat{k}^{-1}(\hat{\Lambda} \circ \pi^*)(t_i)\hat{k}, \hat{\Lambda}(\hat{t}_\infty))_{\hat{\mathcal{D}}} (\hat{k}^{-1}(\hat{\Lambda} \circ \pi^*)(t_j)\hat{k}, \hat{T}^\infty)_{\hat{\mathcal{D}}} \right] = \\ &- \left[T^i \pi(\hat{k}) \otimes T^j \pi(\hat{k}) \right] \times (\pi(\hat{k})^{-1} \Lambda(t_i) \pi(\hat{k}), \Lambda(t_l))_{\mathcal{D}} (\pi(\hat{k})^{-1} \Lambda(t_j) \pi(\hat{k}), T^l)_{\mathcal{D}}. \end{aligned} \quad (5.138)$$

Now we use the formulas (5.88) and (5.137) to calculate the $\{.,.\}_G$ -contribution to the Poisson bracket $\{.,.\}_{qWZ}$ according to the decomposition (5.135). The result is

$$\begin{aligned} \{k \otimes k\}_G^R &= \\ &= -(T^i k \otimes T^j k) \times (k^{-1} \Lambda(t_i) k, \Lambda(t_l))_{\mathcal{D}} (k^{-1} \Lambda(t_j) k, T^l)_{\mathcal{D}} = [(k \otimes k), (T^i \otimes \Lambda(t_i))], \end{aligned} \quad (5.139)$$

which is by the way nothing but the Poisson-Lie bracket on $G = LG_0$ induced by the double $D = LG_0^{\mathbb{C}}$. The calculation giving this result follows step-by-step the computation leading from (5.50) to (5.52).

Although the continuation of this affine story is very similar to the finite dimensional case described in section 5.1.3., some additional care is needed in the affine case because of the infinite number of the elements of the basis $(T^i, \Lambda(t_i))$ of $\mathcal{D} = L\mathcal{G}_0$. Indeed, we must give the meaning to the series $T^i \otimes \Lambda(t_i)$. Recall that the basis is given by (cf. (5.116) and (5.119)).

$$T^i = T^\mu, B^{\hat{\alpha}}, C^{\hat{\alpha}}, \quad \Lambda(t_i) = \varepsilon H^\mu, \frac{\varepsilon |\hat{\alpha}|^2}{\sqrt{2}} E^{\hat{\alpha}}, -i \frac{\varepsilon |\hat{\alpha}|^2}{\sqrt{2}} E^{\hat{\alpha}}, \quad \hat{\alpha} \in \hat{\Phi}_+. \quad (5.140)$$

A simple computation then shows that

$$T^i \otimes \Lambda(t_i) = i\varepsilon H^\mu \otimes H^\mu + i\varepsilon \sum_{\hat{\alpha} \in \hat{\Phi}} |\hat{\alpha}|^2 E^{-\hat{\alpha}} \otimes E^{\hat{\alpha}}. \quad (5.141)$$

We wish to calculate this expression in the evaluation representation of $L\mathcal{G}_0$. Recall that its representation space LV_0 is given by square-integrable

maps from the loop circle into the representation space V_0 of some finite-dimensional (typically irreducible) representation of \mathcal{G}_0 . This means (e.g. for the affine root $\hat{\alpha} = (\alpha, n)$) that $E^{\hat{\alpha}}$ is to be viewed as $E^\alpha e^{in\sigma}$, where $E^\alpha \in \text{End}(V_0)$ and $e^{in\sigma}$ is "the multiplication by function" operator in $\text{End}(LV_0)$.

Among the summation over all affine roots, we can consider the sub-summation over $\hat{\alpha} = (\alpha, n)$, where α is kept fixed and n acquires whatever integer value. Then it is easy to see that in the evaluation representation the Fourier series over n diverges. This divergence shows that we have to care about the analytic aspect of working with the infinite dimensional symplectic manifolds. In other words, we have to give a meaning to the divergent Fourier series (5.141). The reader should understand that the prescription associating a well-defined function of σ to the series (5.141) is the part of the definition of our chiral quasitriangular WZW model. Indeed, the resulting functions appear in the definition of the symplectic structure of the model.

Of course, our prescription for summing the divergent series must fulfil some consistency conditions. Among them there is the most important one: the Poisson bracket (5.139) must fulfil the Jacobi identity. On the top of this, we shall require that the resulting function of σ be meromorphic. In this way we shall find ourselves in the standard world of the r -matrices.

It turns out that the prescription fulfilling both conditions exists: it is called the Abel-Poisson summation method and is based on the following observation: The series $\sum_{n>0} (a_n \cos n\sigma + b_n \sin n\sigma)$ becomes (uniformly) convergent if we replace (a_n, b_n) by $(r^n a_n, r^n b_n)$, where $0 \leq r < 1$. Its sum we denote $S_r(\sigma)$. If the limit $\lim_{r \rightarrow 1} S_r(\sigma)$ exists, it is called the Abel-Poisson sum of the original series. For example, we have (cf. p. 83 of [41])

$$\sum_{n>0} (e^{in\sigma} - e^{-in\sigma}) = i \cotg \frac{1}{2} \sigma \quad (5.142)$$

in the Abel-Poisson sense. It is indeed this formula, which permits to compute $T^i \otimes \Lambda(t_i)$ in the evaluation representation:

$$T^i \otimes \Lambda(t_i) \equiv \varepsilon \hat{r}(\sigma - \sigma') = \varepsilon r + \varepsilon C \cotg \frac{1}{2}(\sigma - \sigma'). \quad (5.143)$$

Recall that

$$r = \sum_{\alpha \in \Phi_+} \frac{i|\alpha|^2}{2} (E^{-\alpha} \otimes E^\alpha - E^\alpha \otimes E^{-\alpha}); \quad (5.144)$$

$$C = \sum_{\mu} H^{\mu} \otimes H^{\mu} + \sum_{\alpha \in \Phi_+} \frac{|\alpha|^2}{2} (E^{-\alpha} \otimes E^{\alpha} + E^{\alpha} \otimes E^{-\alpha}). \quad (5.145)$$

Inserting (5.143) into the formula (5.139), we obtain

$$- \{k(\sigma) \otimes k(\sigma')\}_G^R = \varepsilon[\hat{r}, (k(\sigma) \otimes k(\sigma'))]. \quad (5.146)$$

This Poisson bracket obeys indeed the Jacobi identity, since it can be easily checked that the r -matrix $\hat{r}(\sigma - \sigma')$ satisfies the ordinary Yang-Baxter equation with spectral parameter, i.e.

$$[\hat{r}^{12}(\sigma_1 - \sigma_2), \hat{r}^{13}(\sigma_1 - \sigma_3) + \hat{r}^{23}(\sigma_2 - \sigma_3)] + [\hat{r}^{13}(\sigma_1 - \sigma_3), \hat{r}^{23}(\sigma_2 - \sigma_3)] = 0. \quad (5.147)$$

The calculation of the bracket $\{k \otimes k\}_0$ in (5.135) is even more straightforward. We use (5.131) and (5.134) to arrive at

$$\{k \otimes k\}_0 = (k \otimes k) \hat{r}'_{\varepsilon}(\hat{\phi}_{\kappa}), \quad (5.148)$$

where

$$\hat{r}'_{\varepsilon}(\hat{\phi}_{\kappa}) = \sum_{\hat{\alpha} \in \hat{\Phi}_+} \frac{\varepsilon |\hat{\alpha}|^2}{(1 - e^{\frac{\varepsilon}{2} |\hat{\alpha}|^2 \langle i\hat{\alpha}^{\vee}, \hat{\phi}_{\kappa} \rangle})} B^{\hat{\alpha}} \wedge C^{\hat{\alpha}}. \quad (5.149)$$

Recall also that $\hat{\phi}_{\kappa} = \kappa \hat{t}_{\infty} + \kappa a^{\mu} \pi^*(t_{\mu})$. Putting together (5.146) and (5.149), we arrive at the most important formula of this paper:

$$\{k \otimes k\}_{qWZ} = (k \otimes k) \hat{r}_{\varepsilon}(\hat{\phi}_{\kappa}) + \varepsilon \hat{r}(k \otimes k) \quad (5.150)$$

where \hat{r} is the standard affine r -matrix (5.143) and

$$\hat{r}_{\varepsilon}(\hat{\phi}_{\kappa}) = i\varepsilon \sum_{\hat{\alpha} \in \hat{\Phi}} \frac{|\hat{\alpha}|^2}{2} \coth\left(\frac{\varepsilon}{2} |\hat{\alpha}|^2 \langle i\hat{\alpha}^{\vee}, \hat{\phi}_{\kappa} \rangle\right) E^{\hat{\alpha}} \otimes E^{-\hat{\alpha}}. \quad (5.151)$$

Note that here the summation goes over all roots. It is useful to note, that \hat{r}_{ε} is nothing but the direct affinization of the formula (5.60) for r_{ε} . We shall refer to $\hat{r}_{\varepsilon}(\hat{\phi}_{\kappa})$ as to the deformed affine dynamical r -matrix. We shall see in a moment that (5.151) in the evaluation representation will indeed fulfil the dynamical Yang-Baxter equation with spectral parameter.

It is insightful to visualise the σ -dependence in (5.151). Recalling the explicit expressions (3.83), (3.84) for $\langle i\hat{\alpha}^{\vee}, \hat{\phi}_{\kappa} \rangle$, we can write

$$\hat{r}_{\varepsilon}(\hat{\phi}_{\kappa})(\sigma - \sigma') = -i\varepsilon \sum_{\mu, n \neq 0} \coth(-\varepsilon \kappa n) (H^{\mu} \otimes H^{\mu}) e^{in(\sigma - \sigma')}$$

$$-i\varepsilon \sum_{\alpha \in \Phi, n \in \mathbf{Z}} \frac{|\alpha|^2}{2} \coth(-\varepsilon \kappa a^\mu \langle \alpha, H^\mu \rangle - \varepsilon \kappa n) (E^\alpha \otimes E^{-\alpha}) e^{in(\sigma - \sigma')}. \quad (5.152)$$

Of course, the summation is to be taken in the Abel-Poisson sense.

We use the following classical formulae [46]

$$\sigma_{-y}(z, \tau) = \pi(\cotg \pi z + \cotg \pi y) + 4\pi \sum_{m, n > 0} e^{2\pi i \tau m n} \sin 2\pi(mz + ny); \quad (5.153)$$

$$\rho(z, \tau) = \pi \cotg \pi z + 4\pi \sum_{n > 0} \frac{e^{2\pi i n \tau} \sin 2\pi n z}{1 - e^{2\pi i n \tau}}, \quad (5.154)$$

where the functions $\rho(z, \tau), \sigma_w(z, \tau)$ are defined as (cf. [21, 22, 18])

$$\sigma_w(z, \tau) = \frac{\theta_1(w - z, \tau) \theta_1'(0, \tau)}{\theta_1(w, \tau) \theta_1(z, \tau)}, \quad \rho(z, \tau) = \frac{\theta_1'(z, \tau)}{\theta_1(z, \tau)}. \quad (5.155)$$

Note that $\theta_1(z, \tau)$ is the Jacobi theta function¹

$$\theta_1(z, \tau) = - \sum_{j=-\infty}^{\infty} e^{\pi i (j + \frac{1}{2})^2 \tau + 2\pi i (j + \frac{1}{2})(z + \frac{1}{2})}, \quad (5.156)$$

the prime ' means the derivative with respect to the first argument z and the argument τ (the modular parameter) is a nonzero complex number such that $\text{Im } \tau > 0$.

Now we can sum up the Fourier series (5.152) by using the classical formulae (5.153), (5.154) and the relation (5.142). First we obtain²

$$\sum_{n \in \mathbf{Z}} e^{2\pi i z n} (1 + \coth(a + i\pi n \tau)) = -\frac{i}{\pi} \sigma_{\frac{ia}{\pi}}(z, \tau), \quad (5.157)$$

and

$$1 + \sum_{n \in \mathbf{Z} \setminus 0} e^{2\pi i z n} (1 + \coth(i\pi n \tau)) = -\frac{i}{\pi} \rho(z, \tau). \quad (5.158)$$

¹We have $\theta_1(z, \tau) = \vartheta_1(\pi z, \tau)$ with ϑ_1 in [46].

²The formulae (5.157) and (5.158) appear also in [18] but with several misprinted signs. Those wrong signs turn out to be innocent in the context of [18] since they conspire to give an r -matrix which also fulfils the dynamical Yang-Baxter equations (5.163). In fact, the correct and wrong r -matrices differ by the gauge transformation of type 4. (cf. Section 4.2. of [18]).

Now by using (5.157) and (5.158), we finally arrive at

$$\{k(\sigma) \otimes k(\sigma')\}_{qWZ} = (k(\sigma) \otimes k(\sigma'))\hat{r}_\varepsilon(a^\mu, \sigma - \sigma') + \varepsilon\hat{r}(\sigma - \sigma')(k(\sigma) \otimes k(\sigma')), \quad (5.159)$$

where $\hat{r}_\varepsilon(a^\mu, \sigma)$ is the Felder-Wieczerkowski [22] elliptic dynamical r -matrix given by

$$\begin{aligned} \hat{r}_\varepsilon(a^\mu, \sigma) = \\ = -\frac{\varepsilon}{\pi}\rho\left(\frac{\sigma}{2\pi}, \frac{i\kappa\varepsilon}{\pi}\right)H^\mu \otimes H^\mu - \frac{\varepsilon}{\pi} \sum_{\alpha \in \Phi} \frac{|\alpha|^2}{2} \sigma_{\frac{\varepsilon\kappa a^\mu \langle \alpha, H^\mu \rangle}{\pi i}} \left(\frac{\sigma}{2\pi}, \frac{i\kappa\varepsilon}{\pi}\right) E^\alpha \otimes E^{-\alpha}. \end{aligned} \quad (5.160)$$

The (quasitriangular) braiding relation (5.159) plays the same role in the quasitriangular chiral WZW model as the braiding relation (3.99) in the standard chiral WZW theory. The description of the bracket $\{.,.\}_{qWZ}$ on M_L^{WZ} is then completed by the following formula, that can be easily derived from (5.131) and (5.134):

$$\{k, a^\mu\}_{qWZ} = \frac{1}{\kappa} k T^\mu, \quad \{a^\mu, a^\nu\}_{qWZ} = 0. \quad (5.161)$$

It remains to verify that our Abel-Poisson summation method did indeed give the consistent result. First of all, both r -matrices appearing in (5.159) are clearly meromorphic functions of σ . The Jacobi identity for the Poisson brackets (5.159) and (5.161) then requires (5.147) and

$$[\hat{r}_\varepsilon, 1 \otimes T^\mu + T^\mu \otimes 1] \equiv [\hat{r}_\varepsilon^{12}, (T^\mu)^1 + (T^\mu)^2] = 0; \quad (5.162)$$

$$\begin{aligned} & [\hat{r}_\varepsilon^{12}(\sigma_1 - \sigma_2), \hat{r}_\varepsilon^{13}(\sigma_1 - \sigma_3) + \hat{r}_\varepsilon^{23}(\sigma_2 - \sigma_3)] + [\hat{r}_\varepsilon^{13}(\sigma_1 - \sigma_3), \hat{r}_\varepsilon^{23}(\sigma_2 - \sigma_3)] + \\ & + \frac{1}{\kappa} \left(\frac{\partial}{\partial a^\mu} \hat{r}_\varepsilon^{12} \right) (T^\mu)^3 + \frac{1}{\kappa} \left(\frac{\partial}{\partial a^\mu} \hat{r}_\varepsilon^{23} \right) (T^\mu)^1 + \frac{1}{\kappa} \left(\frac{\partial}{\partial a^\mu} \hat{r}_\varepsilon^{31} \right) (T^\mu)^2 = 0. \end{aligned} \quad (5.163)$$

The relation (5.163) is called the dynamical Yang-Baxter equation with spectral parameter [21, 18]. It is straightforward to check, that the elliptic r -matrix $\hat{r}_\varepsilon(a^\mu, \sigma)$ does verify both conditions (5.162) and (5.163).

It is instructive to rewrite the defining q WZW Poisson brackets (5.159) and (5.161) in terms of the monodromic variables $m(\sigma)$ defined by the relation (3.103). The result is

$$\{m(\sigma) \otimes m(\sigma')\}_{qWZ} = (m(\sigma) \otimes m(\sigma')) B_\varepsilon(a^\mu, \sigma - \sigma') + \varepsilon \hat{r}(\sigma - \sigma')(m(\sigma) \otimes m(\sigma')), \quad (5.164)$$

where $B_\varepsilon(a^\mu, \sigma)$ is the quasitriangular braiding matrix generalizing the matrix $B_0(a^\mu, \sigma)$ defined in (3.105) and (3.106). We find it by generalizing the computation (3.105). The result is

$$B_\varepsilon(a^\mu, \sigma) = -\frac{i}{\kappa} \rho\left(\frac{i\sigma}{2\kappa\varepsilon}, \frac{i\pi}{\kappa\varepsilon}\right) H^\mu \otimes H^\mu - \frac{i}{\kappa} \sum_{\alpha \in \Phi} \frac{|\alpha|^2}{2} \sigma_{a^\mu \langle \alpha, H^\mu \rangle} \left(\frac{i\sigma}{2\kappa\varepsilon}, \frac{i\pi}{\kappa\varepsilon}\right) E^\alpha \otimes E^{-\alpha}. \quad (5.165)$$

We observe that the quasitriangular braiding matrix is again given by the Felder r -matrix but with different modular parameter of the elliptic functions. Indeed, we have derived (5.165) by using the following modular identities

$$\sigma_{-\tau y}(z, \tau) = -\frac{1}{\tau} e^{-2\pi i y z} \sigma_y\left(-\frac{z}{\tau}, -\frac{1}{\tau}\right); \quad (5.166)$$

$$\rho(z, \tau) = -\frac{2\pi i z}{\tau} - \frac{1}{\tau} \rho\left(-\frac{z}{\tau}, -\frac{1}{\tau}\right). \quad (5.167)$$

We can conclude this section by saying that the vertex-IRF transformation can be in a sense interpreted as the modular transformation in the deformation parameter.

5.2.5 q -Kac-Moody primary fields

We return for a moment to the symplectic structure of the standard (non-deformed) chiral WZW model and we note that the equations (3.117) and (3.119) imply

$$\{m(\sigma, j_L^x)\}_{WZ} = x(\sigma) m(\sigma). \quad (5.168)$$

Here $m(\sigma)$ was defined in (3.103), $x(\sigma)$ is some element of $L\mathcal{G}_0$ and j_L^x is the corresponding component of the Kac-Moody current.

The Poisson bracket (5.168) plays a very important role in the quantum theory where it becomes the commutator of two quantum fields m and j_L . The former plays the role of the vertex operator and the latter is the Kac-Moody current. The quantized bracket (5.168) then expresses the fact that m is the Kac-Moody primary field, or, in other words, the Kac-Moody tensor operator.

It will be convenient to rewrite (5.168) by using the inverse vertex-IRF transformation (cf. Section 3.2.6). Let j_L be such \mathcal{G} -valued function on M_L^{WZ}

that $j_L^x = (j_L, x)_{\mathcal{G}}$. Then the bracket (5.168) can be rewritten in the matrix form as follows

$$\{k(\sigma) \otimes j_L(\sigma')\}_{WZ} = 2\pi C \delta(\sigma - \sigma')(k(\sigma) \otimes 1), \quad (5.169)$$

where

$$k(\sigma) = m(\sigma) \exp(-a_\mu \Upsilon(t_\mu) \sigma) \quad (5.170)$$

and C is the Casimir element defined in (5.145).

We wish to find the quasitriangular generalization of the relation (5.169). The quantity $k(\sigma)$ keeps its meaning also in the deformed case; however the standard moment map $\hat{\beta}_L(\hat{k}\hat{a}) \in \hat{\mathcal{G}}^*$ defined on \hat{M}_L is to be replaced by the non-Abelian Poisson-Lie moment map $\hat{b}_L(\hat{k}\hat{a}) \in \hat{B}$. Note that the group multiplication $\hat{k}\hat{a}$ (in the argument) then takes place in the Drinfeld double \hat{D} and $\hat{a} = \exp \hat{\Lambda}(\hat{\phi})$, where $\hat{\phi} = a^\infty \hat{t}_\infty + a^\infty a^\mu \pi^*(t_\mu)$ (cf. (5.114)). What is the analogue of j_L ? In fact, j_L can be written as

$$\pi^* j_L = \Upsilon \circ \iota^* \circ \hat{\beta}_L. \quad (5.171)$$

Note that ι^* is the map from $\hat{\mathcal{G}}^* \rightarrow \mathcal{G}^*$ dual to $\iota : \mathcal{G} \rightarrow \hat{\mathcal{G}}$. But we do not have a canonical map from \hat{B} into B because \hat{B} is the non-Abelian group. There are however two natural possibilities exploiting the maps $m_{L,R}$ defined in Section 4.3, Conventions 4.12. Both are equally good to work with and we shall choose m_R . The deformed case analogue of $\iota^* \circ \hat{\beta}_L$ will be therefore the map

$$F'_L = m_R \circ \hat{b}_L. \quad (5.172)$$

It is easy to see that similarly (as in the non-deformed case), $F'_L(\hat{k}\hat{a})$ is the invariant B -valued function on \hat{M}_L with respect to the central circle action. Restriction of this invariant function on the surface $a^\infty = \kappa$ can be interpreted as the B -valued function on the quasitriangular model space M_L^{WZ} which will be denoted as $F_L(k, a^\mu)$. The quasitriangular analogue of $\{k \otimes j_L\}_{WZ}$ will be now the Poisson bracket $\{k \otimes F_L\}_{qWZ}$. We want to calculate this bracket explicitly.

By definition, we have

$$F'_L(\hat{k}, a^\mu, a^\infty = \kappa) = F_L(\pi(\hat{k}), a^\mu). \quad (5.173)$$

We note, moreover, that (cf. Section 5.2.3)

$$a^\infty = m_L^\infty(\hat{k}\hat{a}) = (m^\infty \circ \hat{b}_L)(\hat{k}\hat{a}), \quad (5.174)$$

where the map $m^\infty : \hat{B} \rightarrow \mathbf{R} = \exp \text{Span}(\hat{\Lambda}(\hat{t}_\infty))$ was introduced in Conventions 4.12 of Section 4.3.

The Poisson bivector on \hat{M}_L was denoted as $\hat{\Sigma}_L$. Because $\hat{b}_L(\hat{k}\hat{a})$ is the non-Abelian moment map, we have the relation (cf. (5.30))

$$\hat{\Sigma}_L(., d\hat{b}_L\hat{b}_L^{-1}) = \hat{\Lambda}(\hat{t}_\infty) \otimes \nabla_{\hat{T}^\infty}^L + (\nu_* \circ \Lambda)(t_i) \otimes \nabla_{\iota(T^i)}^L, \quad (5.175)$$

where $(\hat{T}^\infty, \iota(T^i))$ is the basis of $\hat{\mathcal{G}} = \widehat{LG}_0$ and $(\hat{\Lambda}(\hat{t}_\infty), (\nu_* \circ \Lambda)(t_i))$ is the basis of $\hat{\mathcal{B}}$ in the sense of the double \hat{D} . Recall also that $\hat{\Lambda} \circ \pi^* = \nu_* \circ \Lambda$. The vector fields ∇^L correspond to the left action of the group $\hat{G} = \widehat{LG}_0$ on the model space $\hat{M}_L = \hat{G} \times \hat{A}_+$. Because

$$\hat{b}_L = \exp(m_L^\infty \hat{\Lambda}(\hat{t}_\infty)) F'_L = \exp(a^\infty \hat{\Lambda}(\hat{t}_\infty)) F'_L, \quad (5.176)$$

we infer

$$\begin{aligned} \hat{\Sigma}_L(., da^\infty \hat{\Lambda}(\hat{t}_\infty) + e^{a^\infty \hat{\Lambda}(\hat{t}_\infty)} dF'_L (F'_L)^{-1} e^{-a^\infty \hat{\Lambda}(\hat{t}_\infty)}) = \\ = \hat{\Lambda}(\hat{t}_\infty) \otimes \nabla_{\hat{T}^\infty}^L + (\nu_* \circ \Lambda)(t_i) \otimes \nabla_{\iota(T^i)}^L. \end{aligned} \quad (5.177)$$

From this relation, we obtain readily

$$\{\pi(\hat{k}) \otimes F'_L\}_{\hat{M}_L} = T^i \pi(\hat{k}) \otimes e^{-a^\infty \hat{\Lambda}(\hat{t}_\infty)} (\nu_* \circ \Lambda)(t_i) e^{a^\infty \hat{\Lambda}(\hat{t}_\infty)} F'_L. \quad (5.178)$$

By using the relation (5.134) and the fact that $a^\infty = \kappa$, the symplectic reduction is trivially performed to give

$$\{k \otimes F_L\}_{qWZ} = (T^i \otimes {}^{-\kappa}\Lambda(t_i))(k \otimes F_L) \equiv \varepsilon \hat{r}^\kappa(k \otimes F_L). \quad (5.179)$$

The notation means

$${}^{-\kappa}\Lambda(t_i) \equiv e^{-\kappa(-i\varepsilon\partial_\sigma)} \Lambda(t_i) e^{\kappa(-i\varepsilon\partial_\sigma)}. \quad (5.180)$$

Thus we observe the presence of yet another r -matrix in our game. It is instructive to give explicit formulas for the elements ${}^{-\kappa}\Lambda(t_i)$:

$${}^{-\kappa}\Lambda(t_\mu) = \Lambda(t_\mu), \quad {}^{-\kappa}\Lambda(b_{\hat{\alpha}}) = e^{-\varepsilon\kappa n} \Lambda(b_{\hat{\alpha}}), \quad {}^{-\kappa}\Lambda(c_{\hat{\alpha}}) = e^{-\varepsilon\kappa n} \Lambda(c_{\hat{\alpha}}), \quad (5.181)$$

where n is taken from $\hat{\alpha} = (\alpha, n)$ or from $\hat{\alpha} = (\mu, n)$. The σ dependence of the matrix \hat{r}^κ is therefore as follows

$$\hat{r}^\kappa(\sigma - \sigma') = \hat{r}(\sigma - \sigma' - i\varepsilon\kappa) = r + C \cotg \frac{1}{2}(\sigma - \sigma' - i\varepsilon\kappa). \quad (5.182)$$

The formula (5.179) is the principal result of this section. It is the quasitriangular generalization of the standard primary field condition (5.169). Upon the quantization, the relation should express the crucial property that the primary field should be the tensor operator with respect to the q -current algebra.

Remark: The characterization of the tensor operators of certain quantum groups by means of suitable r -matrices was discussed in [11, 12]. Our results fit in spirit in the framework of those references.

5.2.6 q -deformed current algebra

Recall the basic relation (3.114) defining the standard chiral current algebra:

$$\{j_L^x, j_L^y\}_{WZ} = j_L^{[x,y]} + \kappa\rho(x, y). \quad (5.183)$$

Here $x, y \in \mathcal{G} = L\mathcal{G}_0$. Recall that

$$j_L = \kappa\partial_\sigma m m^{-1} = \kappa a^\mu k \Upsilon(t_\mu) k^{-1} + \kappa\partial_\sigma k k^{-1}. \quad (5.184)$$

The basic relation (5.183) can be cast in the following matrix form

$$\{j_L(\sigma) \otimes j_L(\sigma')\}_{WZ} = \pi\delta(\sigma - \sigma')[C, j_L(\sigma) \otimes 1 - 1 \otimes j_L(\sigma')] + \kappa 2\pi C \partial_\sigma \delta(\sigma - \sigma'), \quad (5.185)$$

where C is the Casimir element defined in (5.145).

Our goal is to calculate the quasitriangular analogue of the current commutator (5.185). For this, we have to evaluate the Poisson brackets of the q -currents $\{F_L \otimes F_L\}_{qWZ}$, $\{F_L^\dagger \otimes F_L^\dagger\}_{qWZ}$ and $\{(F_L^\dagger)^{-1} \otimes F_L\}_{qWZ}$. The reason why the knowledge of the first bracket only is not sufficient is the same as in the finite case (cf. the text between (5.68) and (5.69)). The calculation is similar as the one performed in the previous section. We start with the basic relation (5.175)

$$\hat{\Sigma}_L(., d\hat{b}_L \hat{b}_L^{-1}) = \hat{\Lambda}(\hat{t}_\infty) \otimes \nabla_{\hat{T}^\infty}^L + (\nu_* \circ \Lambda)(t_i) \otimes \nabla_{i(T^i)}^L.$$

Suppose that $f(\hat{k})$ is some function invariant with respect to the central circle action. Then we obtain readily (cf. (5.178))

$$\{f(\hat{k}), F'_L\}_{\hat{M}_L} = \nabla_{i(T^i)}^L f(\hat{k}) \times (e^{-a^\infty \hat{\Lambda}(\hat{t}_\infty)} (\nu_* \circ \Lambda)(t_i) e^{a^\infty \hat{\Lambda}(\hat{t}_\infty)} F'_L). \quad (5.186)$$

Now we take for f the function F'_L itself. Since $F'_L = m_R \circ \hat{b}_L$, we can easily calculate the derivative

$$\begin{aligned} \nabla_{\iota(T^i)}^L F'_L &= \\ &= (\hat{b}_L^{-1} \iota(T^i) \hat{b}_L, \iota(T^p))_{\hat{\mathcal{D}}} F'_L(\nu_* \circ \Lambda)(t_p). \end{aligned} \quad (5.187)$$

Thus we obtain

$$\begin{aligned} \{F'_L \otimes F'_L\}_{\hat{M}_L} &= \\ &= F'_L(\nu_* \circ \Lambda)(t_p) \otimes (e^{-a^\infty \hat{\Lambda}(t_\infty)}(\nu_* \circ \Lambda)(t_i) e^{a^\infty \hat{\Lambda}(t_\infty)}) F'_L \times (F_L^{-1}({}^{-a^\infty} T^i) F'_L, T^p)_{\mathcal{D}}. \end{aligned} \quad (5.188)$$

Now we are ready to get down to the Poisson bracket $\{.,.\}_{qWZ}$ on M_L^{WZ} . We obtain

$$\{F_L \otimes F_L\}_{qWZ} = F_L \Lambda(t_p) \otimes {}^{-\kappa} \Lambda(t_i) F_L \times (F_L^{-1}({}^{-\kappa} T^i) F_L, T^p)_{\mathcal{D}}. \quad (5.189)$$

Now we use the obvious relation

$$F_L^{-1}({}^{-\kappa} T^i) F_L = (F_L^{-1}({}^{-\kappa} T^i) F_L, T_p)_{\mathcal{D}} \Lambda(t_p) + (F_L^{-1}({}^{-\kappa} T^i) F_L, \Lambda(t_p))_{\mathcal{D}} T_p \quad (5.190)$$

to rewrite (5.189) in the form

$$\begin{aligned} \{F_L \otimes F_L\}_{qWZ} &= \\ &= ({}^{-\kappa} T^i \otimes {}^{-\kappa} \Lambda(t_i))(F_L \otimes F_L) - F_L T_p \otimes {}^{-\kappa} \Lambda(t_i) F_L \times (F_L^{-1}({}^{-\kappa} T^i) F_L, \Lambda(t_p))_{\mathcal{D}} = \\ &= ({}^{-\kappa} T^i \otimes {}^{-\kappa} \Lambda(t_i))(F_L \otimes F_L) - (F_L \otimes F_L)(T^i \otimes \Lambda(t_i)). \end{aligned} \quad (5.191)$$

By using the concrete properties of the basis $T^i, \Lambda(t_i)$ given by (5.140), it can be easily checked that (5.191) can be rewritten as

$$\{F_L \otimes F_L\}_{qWZ} = \varepsilon[\hat{r}, F_L \otimes F_L]. \quad (5.192)$$

Note that this formula is fully analogous to the relation (5.68) holding in the finite dimensional (non-affine) case.

The case $\{F_L^\dagger \otimes F_L^\dagger\}_{qWZ}$ can be calculated similarly to yield

$$\{F_L^\dagger \otimes F_L^\dagger\}_{qWZ} = -\varepsilon[\hat{r}, F_L^\dagger \otimes F_L^\dagger]. \quad (5.193)$$

It remains to calculate $\{(F_L^\dagger)^{-1} \otimes F_L\}_{qWZ}$. We proceed as before to arrive at the following counterpart of the relation (5.188):

$$\{(F_L^\dagger)^{-1} \otimes F'_L\}_{\hat{M}_L} = -(F_L'^{-1}({}^{-a^\infty} T^i) F'_L, T^p)_{\mathcal{D}} \times$$

$$\times (F_L^\dagger)^{-1} \nu_*((\Lambda(t_p))^\dagger) \otimes (e^{-a^\infty \hat{\Lambda}(t_\infty)}(\nu_* \circ \Lambda)(t_i) e^{a^\infty \hat{\Lambda}(t_\infty)}) F_L' \quad (5.194)$$

Getting down to the Poisson bracket $\{.,.\}_{qWZ}$ on M_L^{WZ} , we obtain

$$\begin{aligned} \{ (F_L^\dagger)^{-1} \otimes F_L \}_{qWZ} &= - (F_L^\dagger)^{-1} (\Lambda(t_p))^\dagger \otimes {}^{-\kappa} \Lambda(t_i) F_L \times ({}^{-\kappa} T^i, F_L T^p F_L^{-1})_{\mathcal{D}} = \\ &= - (F_L^\dagger)^{-1} (\Lambda(t_p))^\dagger \otimes F_L T^p + (F_L^\dagger)^{-1} (\Lambda(t_p))^\dagger \otimes {}^{-\kappa} T^i F_L \times ({}^{-\kappa} \Lambda(t_i), F_L T^p F_L^{-1})_{\mathcal{D}}. \end{aligned} \quad (5.195)$$

Now we use two obvious relations

$$(F_L^{-1} {}^{-\kappa} \Lambda(t^i) F_L)^\dagger = (F_L^{-1} {}^{-\kappa} \Lambda(t^i) F_L, T^p)_{\mathcal{D}} (\Lambda(t_p))^\dagger, \quad ({}^{-\kappa} \Lambda(t_i))^\dagger = {}^{\kappa} ((\Lambda(t_i))^\dagger) \quad (5.196)$$

to rewrite (5.195) in the form

$$\begin{aligned} \{ (F_L^\dagger)^{-1} \otimes F_L \}_{qWZ} &= \\ &= - ((F_L^\dagger)^{-1} \otimes F_L) ((\Lambda(t_p))^\dagger \otimes T^p) + {}^{\kappa} ((\Lambda(t_p))^\dagger) \otimes {}^{-\kappa} T^p ((F_L^\dagger)^{-1} \otimes F_L). \end{aligned} \quad (5.197)$$

Using the explicit form of the base $(T^i, \Lambda(t_i))$, we find easily that $\varepsilon \hat{r} = (\Lambda(t_p))^\dagger \otimes T^p$. Thus we obtain the final form of the third defining Poisson bracket of the q -current algebra:

$$\{ (F_L^\dagger)^{-1} \otimes F_L \}_{qWZ} = \varepsilon \hat{r}^{2\kappa} ((F_L^\dagger)^{-1} \otimes F_L) - ((F_L^\dagger)^{-1} \otimes F_L) \varepsilon \hat{r}, \quad (5.198)$$

where

$$\hat{r}^{2\kappa}(\sigma - \sigma') = \hat{r}(\sigma - \sigma' - 2i\varepsilon\kappa). \quad (5.199)$$

Remark: The Poisson bracket of the type (5.198) resembles the brackets arising in the description of the structure of the so called twisted Heisenberg double of the reference [43]. Although we do not see here any direct connection of [43] with (5.198) (since M_L^{WZ} does not have even the structure of the double), we believe, nevertheless, that there is a deeper reason why similar formulas appear here and in [43]. Most probably, the double $D = LG_0^{\mathbf{C}}$ equipped with the $qWZW$ bracket reduced from $\hat{\hat{D}}$ could be a sort of the real form of the twisted Heisenberg double of [43]. More precisely: there exists a method of generating a new (non-twisted) Heisenberg double D_{real} from an old one $D_{complex} = GB$ if the latter is equipped with an involution. Indeed, D_{real} is simply the subgroup of $D_{complex}$ consisting of the elements which are stable under the involution. There is a condition to fulfil, however, that the restriction of the bilinear form $(.,.)_{\mathcal{D}}$ on $Lie(D_{real})$ must be non-degenerate. Then D_{real} has canonically the structure of the Heisenberg double.

It can be decomposed as $D_0 = G_0 B_0$ where G_0, B_0 play the role of the mutually dual isotropic Poisson-Lie groups and they themselves consist of the elements of G and B stable under the involution. We conjecture that the similar realification can be made also for the twisted doubles and that among the class of the twisted Heisenberg doubles of LG_0 we can find such $D_{complex}$ that the derived twisted double $D_{real} = LG_0^C$ (given by a suitable involution to be specified) is canonically equipped with the $qWZW$ symplectic structure. However, the fact whether the conjecture is true or false does not have direct implications for our $qWZW$ story and we shall not further study possible connections of our formalism with the theory of the twisted Heisenberg doubles.

There already exists in the literature (cf. [42]) the concept of the q -deformed current algebra. We now show that our construction fits in this framework due to Reshetikhin and Semenov-Tian-Shansky [42]. First we consider the (matrix) observable

$$L = F_L F_L^\dagger. \quad (5.200)$$

The defining commutations relations (5.192), (5.193) and (5.198) can be then equivalently written in terms of one relation only:

$$\begin{aligned} \{L(\sigma) \otimes L(\sigma')\}_{qWZ} &= (L(\sigma) \otimes L(\sigma')) \varepsilon \hat{r}(\sigma - \sigma') + \varepsilon \hat{r}(\sigma - \sigma') (L(\sigma) \otimes L(\sigma')) \\ &- (1 \otimes L(\sigma')) \varepsilon \hat{r}(\sigma - \sigma' + 2i\varepsilon\kappa) (L(\sigma) \otimes 1) - (L(\sigma) \otimes 1) \varepsilon \hat{r}(\sigma - \sigma' - 2i\varepsilon\kappa) (1 \otimes L(\sigma')). \end{aligned} \quad (5.201)$$

Our formula (5.201) now exactly coincides with the defining formula of q -current algebra in the version of Reshetikhin and Semenov-Tian-Shansky [42].

Recall that in the undeformed WZW model, the current $j_L(\sigma)$ can be written in terms of the primary field $m(\sigma)$ as follows

$$j_L = \kappa \partial_\sigma m m^{-1}. \quad (5.202)$$

This relation can be called the classical Knizhnik-Zamolodchikov equation [37] since its quantum analogue is nothing but the standard KZ-equation written in the operatorial form [24]. Recall that here we have used the monodromic variables

$$m(\sigma) = k(\sigma) \exp(-a^\mu T^\mu \sigma)$$

for the description of the phase space M_L^{WZ} of the undeformed (and also deformed) chiral WZW model. The quasitriangular analogue of (5.202) can be easily derived from (5.200) and from (5.172) rewritten as

$$F_L = b_L(k(\sigma + i\varepsilon\kappa)e^{\varepsilon\kappa a^\mu H^\mu}). \quad (5.203)$$

The result is simple and esthetically appealing:

$$L(\sigma) = m(\sigma + i\varepsilon\kappa)m^{-1}(\sigma - i\varepsilon\kappa). \quad (5.204)$$

This is the classical version of the q -KZ equation; as expected, it is not differential but rather a difference equation.

Remark: It is an instructive exercise to calculate the Poisson bracket $\{L(\sigma) \otimes L(\sigma')\}_{qWZ}$ starting with the representation (5.204) and using the formula (5.164). In order to arrive at the formula (5.201), one needs to know the (quasi)periodic behaviour of the involved elliptic functions:

$$\sigma_w(z + \tau, \tau) = \sigma_w(z, \tau)e^{2\pi i w}, \quad \sigma_w(z + 1, \tau) = \sigma_w(z, \tau); \quad (5.205)$$

$$\rho(z + \tau, \tau) = \rho(z, \tau) - 2\pi i, \quad \rho(z + 1, \tau) = \rho(z, \tau). \quad (5.206)$$

5.2.7 The limit $q \rightarrow 1$

The symplectic structure of the quasitriangular chiral WZW model is fully described³ by the fundamental quasitriangular braiding relation (5.153)

$$\{m(\sigma) \otimes m(\sigma')\}_{qWZ} = (m(\sigma) \otimes m(\sigma'))B_\varepsilon(a^\mu, \sigma - \sigma') + \varepsilon \hat{r}(\sigma - \sigma')(m(\sigma) \otimes m(\sigma')), \quad (5.164)$$

where $B_\varepsilon(a^\mu, \sigma)$ is the quasitriangular braiding matrix given by

$$\begin{aligned} B_\varepsilon(a^\mu, \sigma) = \\ = -\frac{i}{\kappa}\rho\left(\frac{i\sigma}{2\kappa\varepsilon}, \frac{i\pi}{\kappa\varepsilon}\right)H^\mu \otimes H^\mu - \frac{i}{\kappa} \sum_{\alpha \in \Phi} \frac{|\alpha|^2}{2} \sigma_{a^\mu \langle \alpha, H^\mu \rangle} \left(\frac{i\sigma}{2\kappa\varepsilon}, \frac{i\pi}{\kappa\varepsilon}\right) E^\alpha \otimes E^{-\alpha}. \end{aligned} \quad (5.165)$$

³The description in the nonmonodromic variables $k(\sigma), a^\mu$ is given by the formulae (5.159) and (5.161).

We wish to show that in the limit $\varepsilon \rightarrow 0$, Eq. (5.164) gives

$$\{m(\sigma) \otimes m(\sigma')\}_{(q=1)WZ} = (m(\sigma) \otimes m(\sigma'))B_0(a^\mu, \sigma - \sigma'), \quad (3.105)$$

where

$$B_0(a^\mu, \sigma) = -\frac{\pi}{\kappa} \left[\eta(\sigma)(H^\mu \otimes H^\mu) - i \sum_{\alpha} \frac{|\alpha|^2}{2} \frac{\exp(i\pi\eta(\sigma)\langle\alpha, H^\mu\rangle a^\mu)}{\sin(\pi\langle\alpha, H^\mu\rangle a^\mu)} E^\alpha \otimes E^{-\alpha} \right]. \quad (3.106)$$

Recall that $\eta(\sigma)$ is the function defined by

$$\eta(\sigma) = 2\left[\frac{\sigma}{2\pi}\right] + 1, \quad (5.207)$$

where $[\sigma/2\pi]$ is the largest integer less than or equal to $\frac{\sigma}{2\pi}$.

Now we observe that the term containing \hat{r} in (5.164) disappears in the limit $\varepsilon \rightarrow 0$. Hence it is enough to show

$$\lim_{\varepsilon \rightarrow 0} \sigma_{a^\mu \langle \alpha, H^\mu \rangle} \left(\frac{i\sigma}{2\kappa\varepsilon}, \frac{i\pi}{\kappa\varepsilon} \right) = -\pi \frac{\exp(i\pi\eta(\sigma)\langle\alpha, H^\mu\rangle a^\mu)}{\sin(\pi\langle\alpha, H^\mu\rangle a^\mu)}, \quad (5.208)$$

$$\lim_{\varepsilon \rightarrow 0} \rho \left(\frac{i\sigma}{2\kappa\varepsilon}, \frac{i\pi}{\kappa\varepsilon} \right) = -i\pi\eta(\sigma). \quad (5.209)$$

Using the formulae (5.205) and (5.206) and the fact $\eta(\sigma + 2\pi) = \eta(\sigma) + 2$ we conclude that it is enough to establish the limits (5.208), (5.209) for $\sigma \in [-\pi, \pi]$. Then we can represent the elliptic functions on the l.h.s. by the series (5.153) and (5.154). For $\sigma \in [-\pi, \pi]$, the contributions of the sums over $m, n > 0$ disappear in the $\varepsilon \rightarrow 0$ limit and we can consider only the "cotangents". This gives immediately (5.208) and (5.209). The correct $\varepsilon \rightarrow 0$ limit of the fundamental braiding relation is therefore established.

Our next task is to establish the $\varepsilon \rightarrow 0$ limit of the quasitriangular Hamiltonian

$$H_L^{qWZ} = -\frac{1}{2\kappa} (\phi, \phi)_{\mathcal{G}^*} - (\widetilde{Dres}_k e^{\hat{\Lambda}(\hat{\phi}_\kappa)})^0.$$

We remind that $(\tilde{b})^0 \equiv m^0(\tilde{b})$ and that the map $m^0 : \tilde{B} \rightarrow \mathbf{R}$ was defined in Conventions 4.12 of Section 4.3 as

$$\hat{m}(\tilde{b}) \exp(m^0(\tilde{b})\tilde{\Lambda}(\tilde{t}_0)) = \tilde{b}, \quad (5.210)$$

where $\hat{m}(\tilde{b}) \in \hat{B}$. This decomposition is unambiguous thus defining the function m^0 . Moreover, since the generator $\tilde{\Lambda}(\tilde{t}_0)$ itself depends on ε also m^0 does which we may indicate by the subscript m_ε^0 .

Now we use the same reasoning as in the finite dimensional case of Section 5.1.3 showing that in the $\varepsilon \rightarrow 0$ limit the dressing action becomes the coadjoint one. Thus we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} (m_\varepsilon^0 \circ \widetilde{Dres}_{\hat{k}})(e^{\hat{\Lambda}_\varepsilon(\hat{\phi}_\kappa)}) &= \langle \widetilde{Coad}_{\hat{k}} \hat{\phi}_\kappa, \tilde{T}^0 \rangle = \\ &= \langle \phi, k^{-1} \partial_\sigma k \rangle + \frac{1}{2} \kappa (k^{-1} \partial_\sigma k, k^{-1} \partial_\sigma k)_\mathcal{G}, \end{aligned} \quad (5.211)$$

where k stands for $\pi(\hat{k})$ for brevity and we indicated by the subscript that $\hat{\Lambda}$ depends on ε . From the relation (5.211), we then obtain immediately

$$\lim_{\varepsilon \rightarrow 0} H_L^{qWZ} = -\frac{1}{2\kappa} (\phi, \phi)_{\mathcal{G}^*} - \langle \phi, k^{-1} \partial_\sigma k \rangle - \frac{\kappa}{2} (k^{-1} \partial_\sigma k, k^{-1} \partial_\sigma k)_\mathcal{G} \equiv H_L^{WZ}. \quad (5.212)$$

Thus the standard chiral Hamiltonian (3.92) is indeed recovered in the limit $\varepsilon \rightarrow 0$. We conclude that the quasitriangular chiral WZW model gives in the limit $\varepsilon \rightarrow 0$ (or $q \rightarrow 1$) the standard chiral WZW theory.

We know that the fundamental bracket (5.164) implies the q -current algebra bracket (5.201). Our next task is to show that in the limit $\varepsilon \rightarrow 0$ (or, equivalently, $q \rightarrow 1$), we recover from (5.179) the standard Kac-Moody primary field condition (5.169) and from (5.201) the ordinary current algebra bracket (5.185).

First we rewrite (5.179) in the equivalent way using the q -current $L(\sigma)$ defined in (5.200). We obtain

$$\begin{aligned} \{k(\sigma) \otimes L(\sigma')\}_{qWZ} &= \\ &= \varepsilon \hat{r}(\sigma - \sigma' - i\varepsilon \kappa) (k(\sigma) \otimes L(\sigma')) - (1 \otimes L(\sigma')) \varepsilon \hat{r}(\sigma - \sigma' + i\varepsilon \kappa) (k(\sigma) \otimes 1). \end{aligned} \quad (5.213)$$

From the classical q -KZ equation (5.204), we derive

$$L(\sigma) = 1 + 2i\varepsilon \kappa \partial_\sigma m m^{-1} + O(\varepsilon^2) = 1 + 2i\varepsilon j_L(\sigma) + O(\varepsilon^2). \quad (5.214)$$

Inserting this into (5.213) and using (5.182), we obtain in the lowest order in ε the desired relation (5.169)

$$\{k(\sigma) \otimes j_L(\sigma')\}_{WZ} = 2\pi C \delta(\sigma - \sigma') (k(\sigma) \otimes 1).$$

Here the δ -function was produced as the following limit $\varepsilon \rightarrow 0^+$

$$4\pi i\delta(\sigma - \sigma') = \cotg \frac{1}{2}(\sigma - \sigma' - i0^+) - \cotg \frac{1}{2}(\sigma - \sigma' + i0^+). \quad (5.215)$$

Now we establish the $q \rightarrow 1$ limit of the q -current algebra

$$\begin{aligned} \{L(\sigma) \otimes L(\sigma')\}_{qWZ} &= (L(\sigma) \otimes L(\sigma'))\varepsilon\hat{r}(\sigma - \sigma') + \varepsilon\hat{r}(\sigma - \sigma')(L(\sigma) \otimes L(\sigma')) \\ &- (1 \otimes L(\sigma'))\varepsilon\hat{r}(\sigma - \sigma' + 2i\varepsilon\kappa)(L(\sigma) \otimes 1) - (L(\sigma) \otimes 1)\varepsilon\hat{r}(\sigma - \sigma' - 2i\varepsilon\kappa)(1 \otimes L(\sigma')). \end{aligned} \quad (5.201)$$

Inserting the ε -expansion (5.214) into (5.201), we obtain in the lowest order ε^2 the correct result

$$\{j_L(\sigma) \otimes j_L(\sigma')\}_{WZ} = \pi\delta(\sigma - \sigma')[C, j_L(\sigma) \otimes 1 - 1 \otimes j_L(\sigma')] + 2\pi\kappa C\partial_\sigma\delta(\sigma - \sigma'). \quad (5.185)$$

Here we have needed three formulae:

$$8\pi i\partial_\sigma\delta(\sigma - \sigma') = \frac{1}{\sin^2 \frac{1}{2}(\sigma - \sigma' + i0^+)} - \frac{1}{\sin^2 \frac{1}{2}(\sigma - \sigma' - i0^+)}; \quad (5.216)$$

$$2\pi i\delta(\sigma - \sigma') = \cotg \frac{1}{2}(\sigma - \sigma' - i0^+) - \cotg \frac{1}{2}(\sigma - \sigma'). \quad (5.217)$$

$$2\pi i\delta(\sigma - \sigma') = \cotg \frac{1}{2}(\sigma - \sigma') - \cotg \frac{1}{2}(\sigma - \sigma' + i0^+). \quad (5.218)$$

The relation (5.216) can be obtained by deriving (5.215) and the remaining equalities (5.217) and (5.218) can be proved by using the Plemelj-Sokhotsky formula.

5.2.8 Quasitriangular exact solution

The simplest way for describing the classical solutions of the quasitriangular chiral WZW model consists in using the monodromic variables $m(\sigma)$ (cf. (3.103)) on the phase space M_L^{WZ} . It turns out that the time evolution is the same as in the non-deformed case, i.e.

$$[m(\sigma)](\tau) = m(\sigma - \tau). \quad (5.219)$$

In order to prove that we have to combine the arguments of Sections 3.1.3 and 3.2.6. First of all, the quasitriangular chiral master model on \tilde{G} can be

solved in the same way as its finite-dimensional counterpart (3.14) on G_0 , with the result (in $\tilde{k}, \tilde{\phi}$ variables on the affine model space \tilde{M}_L)

$$\tilde{k}(\tau) = \tilde{k}_0 \exp\left(\frac{-\tilde{\Upsilon}(\tilde{\phi}_0)}{\kappa}\tau\right), \quad \tilde{\phi}(\tau) = \tilde{\phi}_0. \quad (5.220)$$

Here the multiplication is taken in the sense of the group \tilde{G} .

Now this solution can be projected on the (doubly) reduced phase M_L^{WZ} following step-by-step the argumentation between (3.121) and (3.124). The result is then (5.219).

Remark: It may seem astonishing that the set of classical solutions does not change under the q -deformation. We have met the finite dimensional version of this phenomenon already in Section 5.1.4. The solution of the puzzle is the same. It is the symplectic structure on the phase space that gets deformed in such the way that the G -action ceases to be Hamiltonian but becomes Poisson-Lie. This means that the natural dynamical variables of the group theoretical origin will have modified Poisson brackets and, upon the quantization, modified commutation relations. For instance, the field theoretical correlation functions change.

5.2.9 The left-right glueing

Consider the topological direct product $M_L^{WZ} \times M_R^{WZ}$ where $M_L^{WZ} = LG_0 \times \mathcal{A}_+^1$ and $M_R^{WZ} = LG_0 \times \mathcal{A}_-^1$. Recall that $\mathcal{A}_-^1 \equiv -\mathcal{A}_+^1$. The product symplectic structure on $M_L^{WZ} \times M_R^{WZ}$ is given by the symplectic form

$$\omega_{L \times R}^{qWZ} = \omega_L^{qWZ} + \omega_R^{qWZ}. \quad (5.221)$$

The symplectic form ω_R^{qWZ} differs from ω_L^{qWZ} only by the domain of definition of the variables a^μ . The Hamiltonian on $M_L^{WZ} \times M_R^{WZ}$ is given by

$$H_{L \times R}^{qWZ}(k_L, k_R, a_L^\mu, a_R^\mu) = H_L^{qWZ}(k_L, a_L^\mu) + H_R^{qWZ}(k_R, a_R^\mu). \quad (5.222)$$

Now we perform a symplectic reduction by setting

$$a_L^\mu + a_R^\mu = 0. \quad (5.223)$$

We learn from the Poisson brackets (5.161) that the quantities $a_L^\mu + a_R^\mu$ are the moment maps generating the simultaneous right action of the Cartan

generators T^μ on M_L^{WZ} and M_R^{WZ} . The reduction makes sense only if the Hamiltonian $H_{L \times R}^{qWZ}$ is invariant with respect to this \mathbf{T} -action. But this is the case as we see from (1.20) and from the following chain of formulas

$$\begin{aligned} \widetilde{Dres}_{(\hat{k}e^{\eta_\nu T^\nu})}(e^{\hat{\Lambda}_\varepsilon(\hat{\phi}_\kappa)}) &\equiv \tilde{b}_L(\hat{k}e^{\eta_\nu T^\nu} e^{\hat{\Lambda}_\varepsilon(\hat{\phi}_\kappa)}) = \\ &= \tilde{b}_L(\widetilde{Ad}_{(\hat{k}e^{\eta_\nu T^\nu})} e^{\hat{\Lambda}_\varepsilon(\hat{\phi}_\kappa)}) = \tilde{b}_L(\widetilde{Ad}_{\hat{k}} e^{\hat{\Lambda}_\varepsilon(\hat{\phi}_\kappa)}) \equiv \widetilde{Dres}_{\hat{k}}(e^{\hat{\Lambda}_\varepsilon(\hat{\phi}_\kappa)}). \end{aligned} \quad (5.224)$$

Recall in this respect that (5.224) makes sense since \mathbf{T} *can* be embedded as the subgroup into \widehat{LG}_0 .

We conclude that the reduced symplectic form and Hamiltonian live on the reduced phase space $((LG_0 \times LG_0)/\mathbf{T}^{diag}) \times \mathcal{A}_+^1$. This is nothing but the phase space of the standard full left-right WZW model. We shall not make more explicit the full left-right WZW symplectic structure. The corresponding formulas are complicated and not illuminating. Anyway, the canonical quantization of the quasitriangular WZW model will proceed by quantization of its chiral components. This is fortunate, because all important Poisson brackets are written in terms of the collection of r -matrices. These r -matrices appeared already in the literature in different context (e.g. KZB equation [7, 22]) and their quantum counterparts are often known, too [21]. Of course, the latter circumstance makes the quantization task more accessible.

Chapter 6

Conclusions and outlook

The most important result of this paper is the explicit description of the symplectic structure ω_L^{qWZ} and of the Hamiltonian H_L^{qWZ} of the quasitriangular chiral WZW model. The q -deformation of the standard current algebra then emerged in the natural way. We did not compare our q -current algebra with that of Reshetikhin and Semenov-Tian-Shansky [42]. We believe, however, that the former is the sort of the real form of the latter. This belief stems from the fact that the formalism of [42] (and of [43]) use rather the theory of the complex Heisenberg doubles of the form $D = G \times G$ for an appropriate group G . Our doubles \tilde{D}, \hat{D}, D are not of this form but they are probably real forms of such doubles in the sense of the remark in Section 5.2.6. In any case, our q -current algebra is dictated by the choice of the affine Lu-Weinstein-Soibelman double of the Kac-Moody group. It should be stressed, in particular, that our method applies not only for this particular choice if the double D . There may be different doubles D of the loop group LG_0 and if we succeed to construct the WZW double \tilde{D} (in the sense of the Definition 4.10) then our general construction of Section 4.3 again works.

We have achieved more than the results described above in the sense that we have built up the whole approach how to q -deform the WZW model and its derived products. In fact, we have no doubts that our construction should be easily generalizable to the gauged WZW models by performing an appropriate symplectic reduction induced by equating the non-Abelian moment maps to some fixed values. Also the quasitriangular boundary WZW model should be at reach by first constructing the central extension of the

group of segments and then picking up some of its Drinfeld doubles. Finally, the supersymmetrization of the quasitriangular geodesical model should lead also to the SUSY version of the q -WZW theory.

Among other obtained results, we should mention the universal description of the WZW model in terms of the central biextensions. We can thus argue, that the WZW-like models can be constructed not only for the loop group case. The construction of the doubles D , \hat{D} and \tilde{D} of chapter 4 is another result of this article which permitted to deform the most important case: the loop group WZW model. The crucial thing to do was to prove the global decomposability $\tilde{D} = \tilde{G}\tilde{B} = \tilde{B}\tilde{G}$. We have succeeded to do that in the series of lemmas of section 4.4.

The most important open problem is the quantization of the quasitriangular WZW model. As in the standard case, the chiral part of the model should have the Hilbert space consisting of the set of quantized dressing orbits of the Kac-Moody group. Only the orbits with the integrable highest weight should be present (with the multiplicity 1). Those integrable highest weight correspond to those points in the alcove for which the induced symplectic form on the dressing orbit is integral. The quantized dressing orbits should carry the unitary representation of the q -Kac-Moody group. The q -vertex operators should be the q -Kac-Moody tensor operators fulfilling the quantum version of the relation (5.174). They should obey the quantized braiding relation of the type (5.153); the theory of the quantum groupoids [48] should be relevant for this story. It is important that the Poisson brackets of the basic observables are of the r -matrix type. The explicit knowledge of the corresponding quantum R -matrices should considerably facilitate the quantization program. Another related important task would be to find the q -free field representation of the model. Perhaps the results of [39] might be of use in the sl_2 case although a sort of analytic continuation in κ is needed to arrive from their version of the q -current algebra to ours. We expect that further developments of the ideas presented in this article can reveal interesting connections with results of other references, e.g. [16, 25, 26, 27, 32]. The important question is the status of the Virasoro generators. It is remarkable that the Virasoro group does act on the classical chiral phase space M_L^{WZ} for every value of q . For $q = 1$ (or $\varepsilon = 0$) this action is Hamiltonian. For generic q , if the group action is Poisson-Lie (probably in the combined Virasoro-Kac-Moody sense), or even quasi-Poisson, then the q -Virasoro algebra could

be interpreted as the Hopf or quasi-Hopf algebra, respectively. It would be certainly interesting to analyze this issue in more detail.

Chapter 7

Appendices

7.1 The loop group primer

In what follows, G_0 will be always a simple compact connected and simply connected Lie group. A group G of smooth maps from the circle S^1 into G_0 is called the loop group and it is often denoted as $G = LG_0$. The group structure of G is naturally given by the pointwise multiplication, the unit element e is the constant map with value e_0 , where e_0 is the unit element of G_0 .

The Lie algebra \mathcal{G} of G consists of smooth maps from S^1 into \mathcal{G}_0 again with the pointwise commutator. The invariant symmetric nondegenerate bilinear form on \mathcal{G} is given by

$$(\xi, \eta)_{\mathcal{G}} = \frac{1}{2\pi} \int d\sigma (\xi(\sigma), \eta(\sigma))_{\mathcal{G}_0^{\mathbb{C}}}, \quad (7.1)$$

where $(\cdot, \cdot)_{\mathcal{G}_0^{\mathbb{C}}}$ is the standard Killing-Cartan form on $\mathcal{G}_0^{\mathbb{C}}$ renormalized in such a way that the square of the length of the longest root is equal to two. For example, for the Lie algebra $su(N)$, $(\cdot, \cdot)_{\mathcal{G}_0^{\mathbb{C}}}$ is given by the trace taken in the fundamental representation. We shall identify the dual \mathcal{G}^* of \mathcal{G} with \mathcal{G} itself via the bilinear form $(\cdot, \cdot)_{\mathcal{G}}$. We shall call the corresponding map Υ . Thus

$$\langle \alpha, \xi \rangle = (\Upsilon(\alpha), \xi)_{\mathcal{G}}, \quad \alpha \in \mathcal{G}^*, \quad \xi \in \mathcal{G}. \quad (7.2)$$

The construction of the standard central extension of loop groups goes as follows [34]: First one considers a group DG_0 of smooth maps from the unit

$Disc$ (in the complex plane) into G_0 with the usual pointwise multiplication. We can now define an extended group \widehat{DG}_0 whose elements are pairs (f, λ) where $f \in DG_0$ and $\lambda \in U(1)$ and whose multiplication law reads

$$(f_1, \lambda_1)(f_2, \lambda_2) = (f_1 f_2, \lambda_1 \lambda_2 \exp [2\pi i \gamma(f_1, f_2)]). \quad (7.3)$$

Here γ is a real valued 2-cocycle on DG_0 given by

$$\gamma(f_1, f_2) = \frac{1}{8\pi^2} \int_{Disc} (f_1^{-1} df_1 \wedge df_2 f_2^{-1})_{\mathcal{G}_0^C}. \quad (7.4)$$

Consider now a subgroup ∂G of DG_0 consisting of all smooth maps from $Disc$ into G_0 such that their value on the boundary $\partial Disc = S^1$ is the unit element e_0 of G_0 . Any $g \in \partial G$ can be thought of as a map $g : S^2 \rightarrow G_0$ by identifying the boundary S^1 of $Disc$ with the north pole of S^2 . It turns out that there is a homomorphism $\Theta : \partial G \rightarrow \widehat{DG}_0$ defined by

$$\Theta(g) = (g, \exp [-2\pi i C(g)]), \quad (7.5)$$

where

$$C(g) = \frac{1}{24\pi^2} \int_{Ball} (dgg^{-1} \wedge dgg^{-1} \wedge dgg^{-1})_{\mathcal{G}_0^C}. \quad (7.6)$$

Here $Ball$ is the unit ball whose boundary is S^2 and we have extended the map $g : S^2 \rightarrow G_0$ to a map $g : Ball \rightarrow G_0$. It is not immediately obvious that the homomorphism Θ is correctly defined, or, in other words, that $\exp [-2\pi i C(g)]$ does not depend on the extension of g to B . The standard argument of the independence of this term on the extension can be found e.g. in [34]. The demonstration that Θ is indeed a homomorphism is based on the Polyakov-Wiegmann formula [40] which asserts that

$$C(g_1 g_2) = C(g_1) + C(g_2) - \gamma(g_1, g_2). \quad (7.7)$$

The fact that the image $\Theta(\partial G)$ is a normal subgroup in \widehat{DG}_0 follows from the identity

$$C(f g f^{-1}) + \gamma(f, g) + \gamma(f g, f^{-1}) = C(g). \quad (7.8)$$

The latter identity is also the direct consequence of the Polyakov-Wiegmann formula.

The standard central extension \hat{G} of the group $G = LG_0$ is defined as the factor group $\widehat{DG}_0 / \Theta(\partial G)$. This group is a (nontrivial) circle bundle

over the base space $LG_0 = \{g : S^1 \rightarrow G_0 | g \text{ smooth} \}$. The projection π is $(g, \lambda) \rightarrow g|_{S^1}$ and the center of \hat{G} is represented by the pairs $(1, \lambda) \in \widehat{DG}_0$. The projection homomorphism from \widehat{DG}_0 onto \hat{G} will be referred to as \wp .

Now we are going to calculate the following expression

$$\left(\frac{d}{ds}\right)_{s=0} \hat{g} e^{s\hat{x}} \hat{g}^{-1} e^{-s\hat{x}} \in \hat{\mathcal{G}}, \quad (7.9)$$

where $\hat{g} \in \hat{G}$ and $\hat{x} \in Lie(\hat{G}) \equiv \hat{\mathcal{G}}$. From this computation we shall extract two informations: 1) what is the commutator in $\hat{\mathcal{G}}$; 2) what is the explicit form of the adjoint action of \hat{G} on $\hat{\mathcal{G}}$.

Let us choose two representatives $\wp^{-1}\hat{g}$ and $e^{s\wp_*^{-1}\hat{x}}$ in \widehat{DG}_0 of the classes \hat{g} and $e^{s\hat{x}}$ in \hat{G} . It is clear that

$$\hat{g} e^{s\hat{x}} \hat{g}^{-1} e^{-s\hat{x}} = \wp \left((\wp^{-1}\hat{g}) e^{s\wp_*^{-1}\hat{x}} (\wp^{-1}\hat{g})^{-1} e^{-s\wp_*^{-1}\hat{x}} \right). \quad (7.10)$$

We shall therefore first calculate an expression

$$\left(\frac{d}{ds}\right)_{s=0} (\Gamma, \lambda) (e^{s\xi}, e^{s\alpha}) (\Gamma, \lambda)^{-1} (e^{-s\xi}, e^{-s\alpha}), \quad (7.11)$$

where

$$(\Gamma, \lambda) = \wp^{-1}\hat{g}; \quad (\xi, \alpha) = \wp_*^{-1}\hat{x}. \quad (7.12)$$

Using (7.3), we calculate

$$\begin{aligned} & \left(\frac{d}{ds}\right)_{s=0} (\Gamma, \lambda) (e^{s\xi}, e^{s\alpha}) (\Gamma, \lambda)^{-1} (e^{-s\xi}, e^{-s\alpha}) = \\ & = \left(\frac{d}{ds}\right)_{s=0} (\Gamma, 1) (e^{s\xi}, 1) (\Gamma, 1)^{-1} (e^{-s\xi}, 1) \\ & = \left(\frac{d}{ds}\right)_{s=0} (\Gamma e^{s\xi}, \exp \frac{is}{4\pi} \int_{Disc} (\Gamma^{-1} d\Gamma \frown d\xi)_{\mathcal{G}_0^C}) (\Gamma^{-1} e^{-s\xi}, \exp \frac{is}{4\pi} \int_{Disc} (d\Gamma \Gamma^{-1} \frown d\xi)_{\mathcal{G}_0^C}) \\ & = \left(\frac{d}{ds}\right)_{s=0} (\Gamma e^{s\xi} \Gamma^{-1} e^{-s\xi}, \exp \frac{is}{4\pi} \int_{Disc} \{2(\Gamma^{-1} d\Gamma \frown d\xi)_{\mathcal{G}_0^C} + ([\xi, \Gamma^{-1} d\Gamma] \frown \Gamma^{-1} d\Gamma)_{\mathcal{G}_0^C}\}) \\ & = \left(\frac{d}{ds}\right)_{s=0} (\Gamma e^{s\xi} \Gamma^{-1} e^{-s\xi}, \exp \frac{-is}{2\pi} \int_{Disc} d(\Gamma^{-1} d\Gamma, \xi)_{\mathcal{G}_0^C}) = \\ & = (\Gamma \xi \Gamma^{-1} - \xi, -\frac{i}{2\pi} \int_{S^1 = \partial Disc} (\Gamma^{-1} d\Gamma, \xi)_{\mathcal{G}_0^C}). \end{aligned} \quad (7.13)$$

From (7.13), one may derive the Lie algebra commutator in $Lie(\widehat{DG}_0) = DG_0 + i\mathbf{R}$. Indeed, by setting $\Gamma = e^{t\eta}$ and deriving with respect to t at $t = 0$, one obtains

$$[(\eta, i\alpha), (\xi, i\beta)] = ([\eta, \xi], \frac{i}{2\pi} \int_{S^1 \equiv \partial Disc} (\eta, d\xi)_{\mathcal{G}_0^C}), \quad \alpha, \beta \in \mathbf{R}. \quad (7.14)$$

Finally, the commutator in the Lie algebra $\hat{\mathcal{G}} = Lie(\hat{G})$ is given by the same formula as (7.14) but η and ξ are to be considered as elements of $L\mathcal{G}_0$ rather than those of $D\mathcal{G}_0$. We see also that the map $\iota : \mathcal{G} \rightarrow \hat{\mathcal{G}}$ (cf. Section 2.1) is simply given by

$$\iota(\xi) = (\xi, 0). \quad (7.15)$$

Although it should be already clear from (7.14) what is the formula for the cocycle ρ , we nevertheless write it down explicitly:

$$\rho(\eta, \xi) = \frac{1}{2\pi} \int_{S^1} (\eta, d\xi)_{\mathcal{G}_0^C}. \quad (7.16)$$

From (7.13), we obtain also the formula for the adjoint action

$$\hat{g}(\xi, i\beta)\hat{g}^{-1} = (g\xi g^{-1}, i\beta - \frac{i}{2\pi} \int_{S^1} (g^{-1}dg, \xi)_{\mathcal{G}_0^C}), \quad (7.17)$$

where $g = \pi(\hat{g})$.

7.2 Cotangent bundle of a group manifold

Consider a (possibly infinite dimensional) connected Lie group G and its cotangent bundle T^*G .

The points K of T^*G are couples (P_K, F_K) where P_K is a point in G and F_K is a differential 1-form at the point P_K ; in other words $F_K \in T_{P_K}^*G$. We shall equip T^*G with the standard group structure by introducing the following product QK of two elements $Q, K \in T^*G$:

$$P_{QK} = P_Q P_K; \quad F_{QK} = R_{P_K^{-1}}^* F_Q + L_{P_Q^{-1}}^* F_K. \quad (7.18)$$

Here R^* and L^* denote pull-backs with respect to the right and left translation by elements of G , respectively. The inverse element K^{-1} is given by

$$P_{K^{-1}} = P_K^{-1}, \quad F_{K^{-1}} = -R_{P_K}^* L_{P_K}^* F_K. \quad (7.19)$$

The unit element E fulfills

$$P_E = e, \quad F_E = 0, \quad (7.20)$$

where e is the unit element of G .

Remarks:

- 1) The projection on the base $P : T^*G \rightarrow G$ defined by $P(K) = P_K$ is a morphism of groups according to (7.18).
- 2) Upon the trivialization of the cotangent bundle T^*G by right (or left) translations, the group law (7.18) turns out to correspond to the semidirect product of G and its coalgebra \mathcal{G}^* . The latter is viewed as the Abelian group underlying the vector space \mathcal{G}^* and G acts on \mathcal{G}^* by means of the coadjoint action. However, in what follows we shall rather use the formula (7.18) because the trivialization breaks the natural left-right symmetry of the product.

We shall denote the Lie algebra of the group T^*G as \mathcal{D} . Clearly, \mathcal{D} can be written as a semidirect sum of Lie algebras

$$\mathcal{D} = \mathcal{G} + \mathcal{G}^*, \quad (7.21)$$

where \mathcal{G} acts on \mathcal{G}^* in the coadjoint way and \mathcal{G}^* is the *commutative* Lie subalgebra of \mathcal{D} . It turns out, that there exists a non-degenerate symmetric bilinear form on \mathcal{D} defined by

$$(x + x^*, y + y^*)_{\mathcal{D}} = \langle x^*, y \rangle + \langle y^*, x \rangle, \quad x, y \in \mathcal{G}, \quad x^*, y^* \in \mathcal{G}^*. \quad (7.22)$$

Here $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between the Lie coalgebra \mathcal{G}^* and the Lie algebra \mathcal{G} .

It is the matter of a straightforward calculation to see that the form $(\cdot, \cdot)_{\mathcal{D}}$ is moreover invariant and the subalgebras \mathcal{G} and \mathcal{G}^* are both isotropic with respect to this form. Said in formulas:

$$([X, Y], Z)_{\mathcal{D}} + (Y, [X, Z])_{\mathcal{D}} = 0, \quad (\mathcal{G}, \mathcal{G})_{\mathcal{D}} = (\mathcal{G}^*, \mathcal{G}^*)_{\mathcal{D}} = 0, \quad (7.23)$$

where $X, Y, Z \in \mathcal{D}$.

The cotangent bundle of any manifold M is equipped with the canonical symplectic structure. The corresponding symplectic 2-form ω can be written as

$$\omega = d\theta, \quad (7.24)$$

where θ is a 1-form on T^*M called the symplectic potential. It is defined in a point $K = (P_K, F_K) \in T^*M$ as

$$\theta = P^*F_K. \quad (7.25)$$

In words, θ is the pullback of the form F_K living in $P_K \in M$ by the projection map $P : K \rightarrow P_K$.

It is now convenient to introduce a set of differential operators (vector fields) on T^*G . Define (as in [43])

$$\nabla^L : C^\infty(T^*G) \rightarrow C^\infty(T^*G) \otimes \mathcal{D}^*; \quad \nabla^R : C^\infty(T^*G) \rightarrow C^\infty(T^*G) \otimes \mathcal{D}^* \quad (7.26)$$

as follows

$$\langle \nabla^L \phi, \alpha \rangle(K) = \left(\frac{d}{ds} \right)_{s=0} \phi(e^{s\alpha} K), \quad \langle \nabla^R \phi, \alpha \rangle(K) = \left(\frac{d}{ds} \right)_{s=0} \phi(K e^{s\alpha}). \quad (7.27)$$

Here $\alpha \in \mathcal{D}$, $K \in T^*G$ and $\phi \in C^\infty(T^*G)$.

Define also a linear operator $\mathcal{R} : \mathcal{D} \rightarrow \mathcal{D}$ as follows

$$\mathcal{R}(x + x^*) = x - x^*, \quad x \in \mathcal{G}, \quad x^* \in \mathcal{G}^*. \quad (7.28)$$

Paranthetically, this operator is known as the classical R -matrix and the Lie algebra \mathcal{D} is the factorizable Baxter-Lie algebra in the sense of [43]. We have now the following lemma.

Lemma 7.1 (Semenov-Tian-Shansky [44]): The Poisson bracket corresponding to the symplectic form (7.24) on T^*G can be written as follows

$$\{\phi, \psi\}_{T^*G} = \frac{1}{2}(\nabla^L \phi, \mathcal{R}^* \nabla^L \psi)_{\mathcal{D}^*} + \frac{1}{2}(\nabla^R \phi, \mathcal{R}^* \nabla^R \psi)_{\mathcal{D}^*}. \quad (7.29)$$

Here $(.,.)_{\mathcal{D}^*}$ is the bilinear form on the dual of \mathcal{D} induced by the (nondegenerate) bilinear form $(.,.)_{\mathcal{D}}$ and $\mathcal{R}^* : \mathcal{D}^* \rightarrow \mathcal{D}^*$ is the map dual to \mathcal{R} . It might be illuminating to write the bracket (7.29) in some basis T^i, t_i ; $i = 1, \dots, \dim G$ of \mathcal{D} where T^i 's form the basis of \mathcal{G} and t_i 's the corresponding dual basis of \mathcal{G}^* . We obtain

$$\{\phi, \psi\}_{T^*G} = \frac{1}{2} \langle \nabla^L \phi, T^i \rangle \langle \nabla^L \psi, t_i \rangle - \frac{1}{2} \langle \nabla^L \phi, t_i \rangle \langle \nabla^L \psi, T^i \rangle +$$

$$+ \frac{1}{2} \langle \nabla^R \phi, T^i \rangle \langle \nabla^R \psi, t_i \rangle - \frac{1}{2} \langle \nabla^R \phi, t_i \rangle \langle \nabla^R \psi, T^i \rangle, \quad (7.30)$$

where the standard Einstein summation convention is used.

Remark: It is important to note, that the canonical Poisson bracket on T^*G can be written entirely in terms of the Lie group structure of T^*G . This way of writing this bracket stands at the basis of our Poisson-Lie generalization of the standard WZW story.

Proof: First we realize that the left (right) trivialization of the cotangent bundle T^*G gives a diffeomorphism between T^*G and the direct product of two manifolds G and \mathcal{G}^* . In other words, there exist two global decompositions $T^*G = G\mathcal{G}^* = \mathcal{G}^*G$, where \mathcal{G}^* is the fiber of the cotangent bundle at the unit element e of the group G . Thus we may write for each $K \in T^*G$:

$$K = (g_L(K), 0)(e, \beta_R(K)) = (e, \beta_L(K))(g_R(K), 0), \quad (7.31)$$

where

$$g_L(K) = g_R(K) = P_K; \quad (7.32)$$

$$\beta_L(K) = R_{P_K}^* F_K, \quad \beta_R(K) = L_{P_K}^* F_K. \quad (7.33)$$

Here $L_{P_K}^*$ is the pullback map of the differential forms on G with respect to the left translation by the element P_K and similarly $R_{P_K}^*$ is the right pullback. Instead of somewhat cumbersome expressions (7.31), we shall rather write

$$K = g_L(K)\beta_R(K) = \beta_L(K)g_R(K). \quad (7.34)$$

It is then clear that the functions on T^*G of the form $\Phi(P_K)$ and $\Psi(\beta_L(K))$ generate the whole algebra of smooth functions on T^*G hence it is enough to compute the Poisson brackets between them. Even more specially, instead of arbitrary functions Ψ on \mathcal{G}^* it is sufficient to consider linear ones, i.e. the functions $\langle \beta_L, x \rangle$ where $x \in \mathcal{G}$.

For the case of the group manifold there exists a convenient expression for the symplectic potential θ in terms of the invariant Maurer-Cartan forms. Recall their definitions: In what follows, the expression $\lambda_G(\rho_G)$ will denote the left(right)invariant \mathcal{G} -valued Maurer-Cartan form on the group G defined by

$$\lambda_G(X_g) = L_{g^{-1}*} X_g, \quad \rho_G(X_g) = R_{g^{-1}*} X_g, \quad X_g \in T_g G. \quad (7.35)$$

Note that the forms λ_G and ρ_G are often written also as

$$\lambda_G = g^{-1}dg, \quad \rho_G = dgg^{-1}. \quad (7.36)$$

The symplectic potential θ can be then simply expressed in the "coordinates" (β, g) as

$$\theta = \langle \beta_L, dgg^{-1} \rangle. \quad (7.37)$$

Here we have abandoned the subscript R on g , since anyway $g_R = g_L \equiv g$. By using the formula

$$d(dgg^{-1}) = dgg^{-1} \wedge dgg^{-1}$$

for the exterior derivative of the Maurer-Cartan form ρ_G , we have

$$\omega = \langle d\beta_L \wedge dgg^{-1} \rangle + \langle \beta_L, dgg^{-1} \wedge dgg^{-1} \rangle. \quad (7.38)$$

Now pick up a basis $T^i \in \mathcal{G}$ and its dual basis $t_i \in \mathcal{G}^*$. It is also convenient to use a short-hand notation $\langle \beta_L, T^i \rangle \equiv \beta^i$. Then the form ω can be written as

$$\omega = d\beta^i \wedge R_{g^{-1}}^* t_i + \frac{1}{2} \beta^i f_i^{mn} R_{g^{-1}}^* t_m \wedge R_{g^{-1}}^* t_n, \quad (7.39)$$

where f_i^{mn} are the structure constants of the Lie algebra \mathcal{G} . This expression can be readily inverted to give the corresponding Poisson tensor Π :

$$\Pi = \frac{1}{2} \beta^m f_m^{ij} \frac{\partial}{\partial \beta^i} \wedge \frac{\partial}{\partial \beta^j} - \frac{\partial}{\partial \beta^i} \wedge R_{g^*} T^i. \quad (7.40)$$

Since we have that $\langle \nabla_G^L, x \rangle = R_{g^*} x$, for $x \in \mathcal{G}$, we obtain from Π the following Poisson brackets

$$\{\Phi_1(g), \Phi_2(g)\} = 0; \quad (7.41)$$

$$\{\Phi(g), \langle \beta_L, x \rangle\} = \left(\frac{d}{ds} \right)_{s=0} \Phi(e^{sx} g) \equiv \langle \nabla_G^L \Phi, x \rangle; \quad (7.42)$$

$$\{\langle \beta_L, x \rangle, \langle \beta_L, y \rangle\} = \langle \beta_L, [x, y] \rangle. \quad (7.43)$$

We are going to prove now that the same set of the Poisson bracket can be obtained directly from the Semenov-Tian-Shansky formula (7.29).

For two functions $\Phi_{1,2} : G \rightarrow \mathbf{R}$ we calculate

$$\{\Phi_1(P_K), \Phi_2(P_K)\}_{T^*G} = 0. \quad (7.44)$$

This follows from the following fact

$$\langle \nabla^L \Phi_j(P_K), x^* \rangle = \left(\frac{d}{ds} \right)_{s=0} \Phi_j(P_{e^{sx^*}K}) = \left(\frac{d}{ds} \right)_{s=0} \Phi_j(P_K) = 0, \quad j = 1, 2 \quad (7.45)$$

and from its right analogue. Here x^* is an element of \mathcal{G}^* viewed as the element of \mathcal{D} . Typically, x^* is t_i in (7.45).

The bracket of the type $\{\langle \beta_L, x \rangle, \langle \beta_L, y \rangle\}_{T^*G}$ for $x, y \in \mathcal{G}$ is more involved. In order to compute it, we need to prove the following formulas:

$$\langle \nabla^L \langle \beta_R(K), x \rangle, y \rangle = \langle \nabla^R \langle \beta_L(K), x \rangle, y \rangle = 0; \quad (7.46)$$

$$\langle \nabla^L \langle \beta_L(K), x \rangle, y^* \rangle = \langle \nabla^R \langle \beta_R(K), x \rangle, y^* \rangle = \langle y^*, x \rangle; \quad (7.47)$$

$$\langle \nabla^L \langle \beta_R(K), x \rangle, y^* \rangle = \langle y^*, Ad_{P_K} x \rangle; \quad (7.48)$$

$$\langle \nabla^R \langle \beta_L(K), x \rangle, y^* \rangle = \langle y^*, Ad_{P_K^{-1}} x \rangle; \quad (7.49)$$

$$\langle \nabla^L \langle \beta_L(K), x \rangle, y \rangle = \langle \beta_L(K), [x, y] \rangle; \quad (7.50)$$

$$\langle \nabla^R \langle \beta_R(K), x \rangle, y \rangle = -\langle \beta_R(K), [x, y] \rangle; \quad (7.51)$$

where $x, y \in \mathcal{G}$ and $x^*, y^* \in \mathcal{G}^*$. We prove e.g. only the last formula, one can prove the others in full analogy. We have

$$\begin{aligned} \langle \nabla^R \langle \beta_R(K), x \rangle, y \rangle &= \left(\frac{d}{ds} \right)_{s=0} \langle \beta_R(Ke^{sy}), x \rangle = \\ &= \left(\frac{d}{ds} \right)_{s=0} \langle L_{P(Ke^{sy})}^* F_{Ke^{sy}}, x \rangle = \left(\frac{d}{ds} \right)_{s=0} \langle L_{e^{sy}}^* L_{P_K}^* R_{e^{-sy}}^* F_K, x \rangle = \\ &= \left(\frac{d}{ds} \right)_{s=0} \langle \beta_R(K), L_{e^{sy}*} R_{e^{-sy}*} x \rangle = -\langle \beta_R(K), [x, y] \rangle. \end{aligned} \quad (7.52)$$

Now with the help of the formulas (7.46) - (7.51), we calculate directly

$$\{\langle \beta_L(K), x \rangle, \langle \beta_L(K), y \rangle\}_{T^*G} = \langle \beta_L(K), [x, y] \rangle. \quad (7.53)$$

The remaining bracket between $\Phi : G \rightarrow \mathbf{R}$ and $\langle \beta_L, x \rangle$ can be directly evaluated again with the help of the formulas (7.46) - (7.51):

$$\{\Phi(P_K), \langle \beta_L(K), x \rangle\}_{T^*G} = \frac{1}{2} \langle \nabla_G^L \phi, T^i \rangle \langle x, t_i \rangle + \frac{1}{2} \langle \nabla_G^R \phi, T^i \rangle \langle t_i, Ad_{P_K^{-1}} x \rangle =$$

$$= \left(\frac{d}{ds} \right)_{s=0} \Phi(e^{sx} P_K) \equiv \langle \nabla_G^L \Phi, x \rangle. \quad (7.54)$$

Here ∇_G^R is the map from $C^\infty(G)$ into $C^\infty(G) \otimes \mathcal{G}^*$ (cf. (7.26,7.27)). The reader has certainly noticed that if the operators ∇^L and ∇^R appear without an index specifying the group it means that they act on the double D . Otherwise we indicate as above ∇_G^L or ∇_G^R .

We recognize in the Poisson brackets (7.44),(7.53) and (7.54) the brackets (7.41),(7.43) and (7.42) obtained by inverting the symplectic form.

For completeness, we list the brackets involving the right "currents" $\langle \beta_R(K), x \rangle$:

$$\{\langle \beta_R(K), x \rangle, \langle \beta_R(K), y \rangle\}_{T^*G} = -\langle \beta_R(K), [x, y] \rangle; \quad (7.55)$$

$$\{\langle \beta_R(K), x \rangle, \langle \beta_L(K), y \rangle\}_{T^*G} = 0; \quad (7.56)$$

$$\{\Phi(P_K), \langle \beta_R(K), x \rangle\} = \langle \nabla_G^R \Phi, x \rangle. \quad (7.57)$$

The lemma is proved.

#

7.3 The symplectic reduction in the dual language

The function $\langle \hat{\beta}_R(\hat{K}), \hat{T}^\infty \rangle = \langle \hat{\beta}_L(\hat{K}), \hat{T}^\infty \rangle$ defined on $T^*\hat{G}$ is the moment map that generates the central circle action on $T^*\hat{G}$. We can see it from (7.30)

$$\{\phi, \langle \hat{\beta}_R, \hat{T}^\infty \rangle\}_{T^*\hat{G}} = \langle \nabla^R \phi, \hat{T}^\infty \rangle \equiv \left(\frac{d}{ds} \right)_{s=0} \phi(\hat{K} e^{t\hat{T}^\infty}) = \langle \nabla^L \phi, \hat{T}^\infty \rangle \quad (7.58)$$

for every function on $T^*\hat{G}$. The symplectic reduction of Section 2.2.3 can be performed also as follows: first fix a submanifold $M_\kappa(\hat{G})$ of $T^*\hat{G}$ consisting of those points where the moment map $\langle \hat{\beta}_R, \hat{T}^\infty \rangle$ acquires the fixed constant value κ . Note that the same constant κ appears in the definition of the Hamiltonian \hat{H} given by (2.41). Now the central circle action preserves the submanifold $M_\kappa(\hat{G})$, hence we can consider the space of its orbits $M_\kappa(\hat{G})/U(1)$. In topologically good cases the latter is a manifold and, in

fact, it is the reduced phase space. We can easily calculate the reduced Poisson bracket of functions η and ρ living on the reduced symplectic manifold $M_\kappa(\hat{G})/U(1)$. First we take their pullbacks $\Pi^*\eta$ and $\Pi^*\rho$ with respect to the map $\Pi : M_\kappa(\hat{G}) \rightarrow M_\kappa(\hat{G})/U(1)$ that associates to a point in $M_\kappa(\hat{G})$ the corresponding central circle orbit. Those pullbacks are functions living on $M_\kappa(\hat{G})$. We extend them on the whole unreduced manifold $T^*\hat{G}$ in such a way that they be invariant with respect to the central circle action on $T^*\hat{G}$. We compute the unreduced Poisson bracket of the extended functions and we restrict the result on $M_\kappa(\hat{G})$. The function on $M_\kappa(\hat{G})$ thus obtained is clearly invariant with respect to the central circle action (this follows from the Jacobi identity for the Poisson bracket); or, in other words, it is constant on the orbits. It is then a pullback of some function living on $M_\kappa(\hat{G})/U(1)$. The latter is nothing but the reduced Poisson bracket $\{\eta, \rho\}_{red}$.

The mechanism of the symplectic reduction is perhaps even more transparent in the dual language liked by noncommutative geometers. The role of the manifold $T^*\hat{G}$ is played by the algebra \mathcal{A} of smooth functions on $T^*\hat{G}$. The algebra of functions on the reduced phase space $M_\kappa(\hat{G})/U(1)$ can be obtained in two steps. First one considers the subalgebra $Inv(\mathcal{A}) \subset \mathcal{A}$ consisting of functions in \mathcal{A} whose unreduced Poisson bracket with the moment map $\langle \hat{\beta}_L, \hat{T}^\infty \rangle$ vanishes. There is a distinguished ideal $I_\kappa(\mathcal{A})$ in $Inv(\mathcal{A})$ consisting of the functions of the form $Inv(\mathcal{A})(\langle \hat{\beta}_L, \hat{T}^\infty \rangle - \kappa)$. Factorizing $Inv(\mathcal{A})$ by its ideal $I_\kappa(\mathcal{A})$ gives the algebra of functions on the reduced symplectic manifold $M_\kappa(\hat{G})/U(1)$. The reduced Poisson bracket of η and ρ as above can be computed by choosing any representatives of the classes η and ρ in $Inv(\mathcal{A})$ and by computing the unreduced Poisson bracket of those representatives. The last step consists in taking the class of the result.

Of course, the symplectic reduction of some dynamical system is a consistent procedure if the Hamiltonian of the unreduced system (Poisson) commutes with the moment map. In this case the Hamiltonian is an element of $Inv(\mathcal{A})$ and as such it gives rise to some function on the reduced phase space. This function is called the Hamiltonian of the reduced system. In our case we have to show that the Hamiltonian

$$\hat{H}(\hat{K}) = -\frac{1}{2\kappa}(\iota^*(\hat{\beta}_L), \iota^*(\hat{\beta}_L))_{\mathcal{G}^*} - \frac{1}{2\kappa}(\iota^*(\hat{\beta}_R), \iota^*(\hat{\beta}_R))_{\mathcal{G}^*} \quad (2.41)$$

is invariant function with respect to the central circle action on $T^*\hat{G}$. It is

easy to see this since we have from (7.18) and (7.33)

$$\begin{aligned}\hat{\beta}_R(e^{s\hat{T}^\infty}\hat{K}) &= L_{\hat{P}_{(e^{s\hat{T}^\infty}\hat{K})}}^* \hat{F}_{(e^{s\hat{T}^\infty}\hat{K})} = L_{\hat{P}_K}^* L_{e^{s\hat{T}^\infty}}^* \hat{F}_{(e^{s\hat{T}^\infty}\hat{K})} \\ &= L_{\hat{P}_K}^* L_{e^{s\hat{T}^\infty}}^* L_{e^{-s\hat{T}^\infty}}^* \hat{F}_{\hat{K}} = L_{\hat{P}_K}^* \hat{F}_{\hat{K}} = \hat{\beta}_R(\hat{K}).\end{aligned}\quad (7.59)$$

In the same way we may check the invariance of $\hat{\beta}_L$ and hence of the whole Hamiltonian H .

We are going to show that the reduced symplectic manifold $M_\kappa(\hat{G})/U(1)$ is indeed diffeomorphic to the cotangent bundle T^*G of the non-extended group G . On the other hand, the reduced symplectic structure does *not* coincide with the canonical symplectic structure on T^*G (unless $\kappa = 0$).

7.3.1 The map between $M_\kappa(\hat{G})/U(1)$ and T^*G .

There is a distinguished subgroup of $T^*\hat{G}$ that we shall denote $M_0(\hat{G})$. It is formed by those elements \hat{K} of $T^*\hat{G}$ that satisfy

$$\hat{F}_{\hat{K}}(L_{\hat{P}_{\hat{K}}^*}\hat{T}^\infty) = 0 \quad (7.60)$$

or, equivalently,

$$\langle \hat{\beta}_R(\hat{K}), \hat{T}^\infty \rangle = 0. \quad (7.61)$$

Here the hats indicate that we are dealing with the extended group \hat{G} . The map $\hat{\beta}_R : \hat{G} \rightarrow \hat{\mathcal{G}}^*$ is defined as in (7.33). L_* is the push-forward map acting on the vector \hat{T}^∞ which lives in the Lie algebra $\hat{\mathcal{G}}$ of \hat{G} viewed as the tangent space $T_{\hat{e}}\hat{G}$ at the unit element \hat{e} of \hat{G} . The reader may note that the condition (7.60) is equivalent to $\hat{F}_{\hat{K}}(R_{\hat{P}_{\hat{K}}^*}\hat{T}^\infty) = 0$. This follows from the fact that the vector $L_{\hat{P}_{\hat{K}}^*}\hat{T}^\infty$ corresponds to the *right* infinitesimal action of the central circle (injected in \hat{G}) at the point $\hat{P}_{\hat{K}}$ and from the fact that the extension is central hence the left action of the $U(1)$ coincides with the right action.

It is the matter of a direct check that the elements \hat{K} fulfilling (7.60), (7.61) form a subgroup of $T^*\hat{G}$. We shall now show that $M_0(\hat{G})$ is naturally the central extension of the group T^*G by the circle group $U(1)$. In order to write down the corresponding exact sequence of group homomorphisms

$$1 \rightarrow U(1) \rightarrow M_0(\hat{G}) \xrightarrow{\Pi_0} T^*G \rightarrow 1, \quad (7.62)$$

we have to specify the injection of $U(1)$ into $M_0(\hat{G})$ and the homomorphism $\Pi : M_0(\hat{G}) \rightarrow T^*G$. The injection is clear since \hat{G} is the subgroup of $M_0(\hat{G})$ (it is formed by the elements \hat{K} of $T^*(\hat{G})$ with $\hat{F}_{\hat{K}} = 0$) hence we inject $U(1)$ in \hat{G} as in (2.1) and this trivially induces the injection in (7.62).

Now the homomorphism Π_0 is constructed as follows: one first notes that the kernel of the push-forward map π_* at some point $\hat{P}_{\hat{K}} \in \hat{G}$ is linearly generated precisely by the vector $L_{\hat{P}_{\hat{K}}}^* \hat{T}^\infty (= R_{\hat{P}_{\hat{K}}}^* \hat{T}^\infty)$. In other words, the condition (7.60) implies that there exists a unique 1-form F_K living in the point $\pi(\hat{P}_{\hat{K}})$ such that

$$\pi^* F_K = \hat{F}_{\hat{K}}. \quad (7.63)$$

Rephrasing differently, every form $\hat{F}_{\hat{K}}$, living in the point $\hat{P}_{\hat{K}}$ and satisfying the condition (7.60), is a pull-back by π of the uniquely given form F_K in the point $\pi(\hat{P}_{\hat{K}})$. But a form in a point defines an element of the group T^*G ; we have denoted it as K in our context. Now the map Π_0 is defined by associating to every $\hat{K} \in M_0(\hat{G})$ the corresponding element $K \in T^*G$. Another way of writing this is as follows

$$P_{\Pi_0(\hat{K})} = \pi(\hat{P}_{\hat{K}}); \quad (7.64)$$

$$\hat{F}_{\hat{K}} = \pi^* F_{\Pi_0(\hat{K})}. \quad (7.65)$$

Lemma 7.2: Π_0 is the group homomorphism.

Proof: One has to verify the validity of two relations:

$$P(\Pi_0(\hat{Q}\hat{K})) = P(\Pi_0(\hat{Q}))P(\Pi_0(\hat{K})); \quad (7.66)$$

$$F_{\Pi_0(\hat{Q}\hat{K})} = R_{\Pi_0(\hat{K})}^{*-1} F_{\Pi_0(\hat{Q})} + L_{\Pi_0(\hat{Q})}^{*-1} F_{\Pi_0(\hat{K})}. \quad (7.67)$$

The first one (7.66) is simple, one has from (7.64)

$$P(\Pi_0(\hat{Q}\hat{K})) = \pi(\hat{P}(\hat{Q}\hat{K})) = \pi(\hat{P}(\hat{Q}))\pi(\hat{P}(\hat{K})) = P(\Pi_0(\hat{Q}))P(\Pi_0(\hat{K})). \quad (7.68)$$

To prove the second one, we have from (7.65) and from the "hat"-version of (7.18)

$$\pi^* F_{\Pi_0(\hat{Q}\hat{K})} = \hat{F}_{\hat{Q}\hat{K}} = R_{\hat{P}_{\hat{K}}}^{*-1} \pi^* F_{\Pi_0(\hat{Q})} + L_{\hat{P}_{\hat{Q}}}^{*-1} \pi^* F_{\Pi_0(\hat{K})}. \quad (7.69)$$

From the homomorphism property of $\pi : \hat{G} \rightarrow G$, it follows for every $\hat{k} \in \hat{G}$:

$$\pi^* R_{\pi(\hat{k})}^* = R_{\hat{k}}^* \pi^* \quad (7.70)$$

and similarly for the left translation by \hat{k} . Combining this fact together with (7.64) and (7.65) we arrive at

$$\pi^* F_{\Pi_0(\hat{Q}\hat{K})} = \pi^* R_{P_{\Pi_0(\hat{K})}^{-1}}^* F_{\Pi_0(\hat{Q})} + \pi^* L_{P_{\Pi_0(\hat{Q})}^{-1}}^* F_{\Pi_0(\hat{K})}. \quad (7.71)$$

But this implies the relation (7.67) because the pull-back π^* of a non-zero form on G would be a non-zero form on \hat{G} .

We shall now again consider submanifold $M_\kappa(\hat{G})$ of the group manifold $T^*\hat{G}$ formed by all points $\hat{K} \in T^*\hat{G}$ fulfilling the condition

$$\hat{F}_{\hat{K}}(L_{\hat{P}_{\hat{K}}^*} \hat{T}^\infty) = \langle \hat{\beta}_R(\hat{K}), \hat{T}^\infty \rangle = \kappa. \quad (7.72)$$

For $\kappa = 0$ we recover from $M_\kappa(\hat{G})$ the manifold $M_0(\hat{G})$ defined above, hence our notation is consistent. Note, however, that if $\kappa \neq 0$, the manifold $M_\kappa(\hat{G})$ is not a subgroup of $T^*\hat{G}$.

We can construct a natural diffeomorphism relating $M_\kappa(\hat{G})$ and $M_0(\hat{G})$. We proceed as follows: first consider a one-parameter subgroup of $T^*\hat{G}$ consisting of those points $\hat{N}(s) \in T^*\hat{G}$ that fulfil

$$\hat{P}_{\hat{N}(s)} = \hat{e}, \quad \hat{F}_{\hat{N}(s)} = s\hat{t}_\infty, \quad (7.73)$$

where $s \in \mathbf{R}$ and \hat{t}_∞ is the 1-form in \hat{e} satisfying

$$\hat{t}_\infty(\hat{T}^\infty) = 1, \quad \hat{t}_\infty(\iota(\mathcal{G})) = 0. \quad (7.74)$$

The conditions (7.74) determine \hat{t}_∞ unambiguously.

Remark: It is easy to check that the group $\hat{N}(s)$ normalizes the group $M_0(\hat{G})$ (i.e. $\hat{N}M_0(\hat{G})\hat{N}^{-1} \subset M_0(\hat{G})$). This means in our context that $T^*\hat{G}$ is naturally a semidirect product of \hat{N} and $M_0(\hat{G})$.

The diffeomorphism relating $M_\kappa(\hat{G})$ and $M_0(\hat{G})$ is now simply given by

$$\hat{K}_0 \rightarrow \hat{N}(\kappa)\hat{K}_0, \quad \hat{K}_0 \in M_0(\hat{G}). \quad (7.75)$$

It is evident that this diffeomorphism commutes with the central circle action hence the spaces of the $U(1)$ orbits on $M_\kappa(\hat{G})$ and on $M_0(\hat{G})$ are also diffeomorphic. Finally, from the exact sequence (7.62) it follows that the space of orbits of $M_0(\hat{G})$ is nothing but T^*G .

7.3.2 The reduced Poisson bracket on $M_\kappa(\hat{G})/U(1)$.

Let us now compute the reduced Poisson bracket on $M_\kappa(\hat{G})/U(1)$. Since the latter is diffeomorphic to T^*G , it is sufficient to determine the Poisson brackets of a distinguished set of functions on T^*G of the form $\Phi(P_K)$ and $\langle\beta_R(K), x\rangle$. Here $\Phi : G \rightarrow \mathbf{R}$, $K \in T^*G$ and $x \in \mathcal{G}$. The functions of this special form generate (via the diffeomorphism above) the whole algebra of functions on $M_\kappa(\hat{G})/U(1)$. Recall that the *canonical* Poisson brackets of those functions are given by the equations (7.41-43)). However, the reduced Poisson bracket of the same quantities are different as the following theorem says:

Theorem 7.3: The reduced symplectic structure on $M_\kappa(\hat{G})/U(1) \cong T^*G$ is fully determined by the following Poisson brackets:

$$\{\Phi_1(P_K), \Phi_2(P_K)\}_{red} = 0; \quad (7.76)$$

$$\{\Phi(P_K), \langle\beta_L(K), x\rangle\}_{red} = \left(\frac{d}{ds}\right)_{s=0} \Phi(e^{sx}P_K) \equiv \langle\nabla_G^L \Phi, x\rangle; \quad (7.77)$$

$$\{\langle\beta_L(K), x\rangle, \langle\beta_L(K), y\rangle\}_{red} = \langle\beta_L(K), [x, y]\rangle + \kappa\rho(x, y). \quad (7.78)$$

Recall that $\rho(x, y)$ is the cocycle characterizing the central extension (2.2).

Remarks: 1) the reduced brackets (7.76) and (7.77) are the same as the canonical brackets (7.41) and (7.42), respectively. However, the reduced bracket (7.78) differs from the corresponding canonical bracket by the cocycle term. We thus see that (unless $\kappa = 0$) the reduced symplectic structure does *not* coincide with the canonical one on T^*G .

2) It may seem that the adding of the cocycle in (7.78) does not represent a "big" change of the Poisson bracket. However, the reader may check as an exercise that the reduced bracket of the type $\{\langle\beta_R, x\rangle, \langle\beta_R, y\rangle\}_{red}$ is already much more complicated than its canonical counterpart (7.55) (We do not list this calculation here since we shall not need it in this paper). The reason why the bracket of the left currents β_L is simpler than that of the right currents β_R is due to the left-right asymmetry in the map relating $M_\kappa(\hat{G})/U(1)$ and T^*G . The choice of this map is not canonical; we have chosen it in this way in order to make contact with the standard description of the WZW symplectic structure in [6]. If we change this map (it is also

possible to make a left-right symmetric choice) the reduced Poisson brackets of the left and right currents $\beta_{L,R}$ change but the theory does not change. Indeed, $\beta_{L,R}$ would then correspond to *different* functions on $M_\kappa(\hat{G})/U(1)$ if the diffeomorphism has changed. We stress that the natural dynamical variables of the problem are $\hat{\beta}_{L,R}$ and they get expressed differently in terms of the observables on T^*G under the change of the diffeomorphism.

3) It does not follow from anywhere that the level κ must be an integer. This constraint will appear at the quantum level. It can be simply understood intuitively, since k is the "momentum" conjugated to the *angle* variable parametrizing the central circle.

In order to prove the theorem, we shall need the following lemma

Lemma 7.4: The pullback $\Pi^*\langle\beta_L, x\rangle$ of the function $\langle\beta_L, x\rangle$ on T^*G via the map $\Pi : M_\kappa(\hat{G}) \rightarrow M_\kappa(\hat{G})/U(1)$ is given by the following formula

$$(\Pi^*\langle\beta_L, x\rangle)(\hat{K}) = \langle\hat{\beta}_L(\hat{K}), \iota(x)\rangle, \quad \hat{K} \in M_\kappa(\hat{G}), \quad x \in \mathcal{G}. \quad (7.79)$$

Remark: The map $\Pi : M_\kappa(\hat{G}) \rightarrow M_\kappa(\hat{G})/U(1)$ has been defined at the beginning of the section 7.3; the statement of the lemma makes sense since we have established in section 7.3.1 that $M_\kappa(\hat{G})/U(1)$ and T^*G are diffeomorphic.

Proof: First we have to prove two important relations:

$$\langle\beta_L(\Pi_0(\hat{K})), x\rangle = \langle\hat{\beta}_L(\hat{K}), \iota(x)\rangle, \quad \hat{K} \in M_0(\hat{G}) \quad (7.80)$$

and

$$\langle\hat{\beta}_L(\hat{N}(s)\hat{K}), \iota(x)\rangle = \langle\hat{\beta}_L(\hat{K}), \iota(x)\rangle, \quad s \in \mathbf{R}, \quad \hat{K} \in T^*\hat{G}, \quad x \in \mathcal{G}. \quad (7.81)$$

Indeed, the first relation is implied by (7.33),(2.3),(7.64),(7.70) and (7.65) as follows

$$\begin{aligned} \langle\beta_L(\Pi_0(\hat{K})), x\rangle &= \langle R_{P_{\Pi_0(\hat{K})}}^* F_{\Pi_0(\hat{K})}, \pi_*(\iota(x))\rangle = \langle F_{\Pi_0(\hat{K})}, R_{\pi(\hat{P}_{\hat{K}})^*} \pi_*(\iota(x))\rangle = \\ &= \langle F_{\Pi_0(\hat{K})}, \pi_* R_{\hat{P}_{\hat{K}}^*} \iota(x)\rangle = \langle \pi^* F_{\Pi_0(\hat{K})}, R_{\hat{P}_{\hat{K}}^*} \iota(x)\rangle = \\ &= \langle \hat{F}_{\hat{K}}, R_{\hat{P}_{\hat{K}}^*} \iota(x)\rangle = \langle R_{\hat{P}_{\hat{K}}}^* \hat{F}_{\hat{K}}, \iota(x)\rangle = \langle \hat{\beta}_L(\hat{K}), \iota(x)\rangle. \end{aligned} \quad (7.82)$$

The second relation in turn follows from (7.18), (7.33) and (7.74):

$$\begin{aligned} \langle \hat{\beta}_L(\hat{N}(s)\hat{K}), \iota(x) \rangle &= \langle R_{\hat{P}_{\hat{N}(s)\hat{K}}}^* \hat{F}_{\hat{N}(s)\hat{K}}, \iota(x) \rangle = \\ &= \langle R_{\hat{P}_{\hat{K}}}^* (sR_{\hat{P}_{\hat{K}}}^* \hat{t}_\infty + \hat{F}_{\hat{K}}), \iota(x) \rangle = s\langle \hat{t}_\infty, \iota(x) \rangle + \langle \hat{\beta}_L(\hat{K}), \iota(x) \rangle = \langle \hat{\beta}_L(\hat{K}), \iota(x) \rangle. \end{aligned} \quad (7.83)$$

Now we have by definition

$$(\Pi^* \langle \beta_L, x \rangle)(\hat{K}) = \langle \beta_L(\Pi_0(\hat{N}(-\kappa)\hat{K})), \iota(x) \rangle. \quad (7.84)$$

Combining (7.80) and (7.81), it follows

$$(\Pi^* \langle \beta_L, x \rangle)(\hat{K}) = \langle \hat{\beta}_L(\hat{K}), \iota(x) \rangle. \quad (7.85)$$

The lemma is proved. #

Proof of the theorem 7.3: Consider now an arbitrary pair of functions ϕ, ψ on T^*G . In order to calculate their reduced Poisson brackets, we first have to pull back them to functions on $M_\kappa(\hat{G})$ via our map $\Pi : M_\kappa(\hat{G}) \rightarrow M_\kappa(\hat{G})/U(1)$. We have, for instance,

$$(\Pi^* \phi)(\hat{K}) = \phi(\Pi_0(\hat{N}(-\kappa)\hat{K})), \quad \hat{K} \in M_\kappa(\hat{G}). \quad (7.86)$$

Now we have to extend the functions $\Pi^* \phi$ and $\Pi^* \psi$ to the whole manifold $T^*\hat{G}$ in such a way that they be invariant with respect to the central circle action. The resulting functions can be referred to as ϕ_{ext} , ψ_{ext} and can be conveniently chosen as

$$\phi_{ext}(\hat{K}) = \phi(\Pi_0(\hat{N}(-\langle \beta_R(\hat{K}), \hat{T}^\infty \rangle)\hat{K})), \quad \hat{K} \in T^*\hat{G} \quad (7.87)$$

and in the same way for ψ_{ext} . Now we should compute the unreduced canonical bracket of ϕ_{ext} and ψ_{ext} .

First we calculate the extensions of two particular functions on T^*G . First one is $(\Phi \circ P)(K) \equiv \Phi(P_K)$ where $\Phi : G \rightarrow \mathbf{R}$ and the second is $\langle \beta_L(K), x \rangle$ where $x \in \mathcal{G}$. We obtain from (7.64) and (7.87)

$$(\Phi \circ P)_{ext}(\hat{K}) = \Phi(P_{\Pi_0(\hat{N}(-\langle \beta_R(\hat{K}), \hat{T}^\infty \rangle)\hat{K})}) =$$

$$= \Phi(\pi(\hat{P}_{\hat{N}(-\langle\beta_R(\hat{K}), \hat{T}^\infty\rangle)\hat{K}})) = \Phi(\pi(\hat{P}_{\hat{K}})) \quad (7.88)$$

and from (7.80), (7.81) and (7.87)

$$\begin{aligned} \langle\beta_L, x\rangle_{ext}(\hat{K}) &= \langle\beta_L(\Pi_0(\hat{N}(-\langle\beta_R(\hat{K}), \hat{T}^\infty\rangle)\hat{K}), x\rangle = \\ &= \langle\hat{\beta}_L(\hat{N}(-\langle\beta_R(\hat{K}), \hat{T}^\infty\rangle)\hat{K}), \iota(x)\rangle = \langle\hat{\beta}_L(\hat{K}), \iota(x)\rangle. \end{aligned} \quad (7.89)$$

Now it is easy to calculate the unreduced bracket $\{\langle\beta_L, x\rangle_{ext}, \langle\beta_L, y\rangle_{ext}\}_{T^*\hat{G}}$. We have from (7.89), (7.43) and (2.7)

$$\begin{aligned} \{\langle\beta_L, x\rangle_{ext}, \langle\beta_L, y\rangle_{ext}\}_{T^*\hat{G}}(\hat{K}) &= \{\langle\hat{\beta}_L(\hat{K}), \iota(x)\rangle, \langle\hat{\beta}_L(\hat{K}), \iota(y)\rangle\}_{T^*\hat{G}} = \\ &= \langle\hat{\beta}_L(\hat{K}), [\iota(x), \iota(y)]\rangle = \langle\hat{\beta}_L(\hat{K}), \iota([x, y])\rangle + \rho(x, y)\hat{T}^\infty = \\ &= \langle\hat{\beta}_L(\hat{K}), \iota([x, y])\rangle + \rho(x, y)\langle\hat{\beta}_L(\hat{K}), \hat{T}^\infty\rangle. \end{aligned} \quad (7.90)$$

The resulting function is clearly invariant with respect to the central circle action. From Lemma 7.4 and Eq.(7.72) it follows that its restriction to $M_\kappa(\hat{G})$ is given by the following expression

$$\langle\hat{\beta}_L(\hat{K}), \iota([x, y])\rangle + \kappa\rho(x, y) = (\Pi^*\langle\beta_l, [x, y]\rangle)(\hat{K}) + \kappa\rho(x, y), \quad \hat{K} \in M_\kappa(\hat{G}). \quad (7.91)$$

Hence we obtain for the reduced Poisson bracket

$$\{\langle\beta_L(K), x\rangle, \langle\beta_L(K), y\rangle\}_{red} = \langle\beta_L(K), [x, y]\rangle + \kappa\rho(x, y). \quad (7.92)$$

This is precisely the relation (7.78).

To prove (7.76) is very easy because from (7.88) and (7.44) we have immediately

$$\{(\Phi_1 \circ P)_{ext}(\hat{K}), (\Phi_2 \circ P)_{ext}(\hat{K})\}_{T^*\hat{G}} = \{\Phi_1(\pi(\hat{P}_{\hat{K}})), \Phi_2(\pi(\hat{P}_{\hat{K}}))\}_{T^*\hat{G}} = 0. \quad (7.93)$$

It remains to verify the relation (7.77). It follows directly from the following computation (cf. (7.42))

$$\begin{aligned} \{(\Phi \circ P)_{ext}(\hat{K}), \langle\beta_L, x\rangle_{ext}(\hat{K})\}_{T^*\hat{G}} &= \{\Phi(\pi(\hat{P}_{\hat{K}})), \langle\hat{\beta}_L(\hat{K}), \iota(x)\rangle\}_{T^*\hat{G}} = \\ &= \left(\frac{d}{ds}\right)_{s=0} \Phi(\pi(e^{s\iota(x)}\hat{P}_{\hat{K}})) = \left(\frac{d}{ds}\right)_{s=0} \Phi(e^{sx}\pi(\hat{P}_{\hat{K}})). \end{aligned} \quad (7.94)$$

The theorem is proved.

#

7.4 Proof of the Lemma 5.8

With the notation of the section 5.2.2, the proposition to be proved reads

Lemma: It holds

$$\hat{m}_L(\widehat{Ad_{\hat{k}}\hat{a}}) = \hat{b}_L(\widehat{Ad_{\hat{k}}\hat{a}}), \quad \tilde{g}_R(\widehat{Ad_{\hat{k}}\hat{a}}) = \hat{g}_R(\widehat{Ad_{\hat{k}}\hat{a}}). \quad (7.95)$$

Proof: Consider any element $(\bar{\bar{l}}, \lambda) \in \widehat{DG_0^C}$. The elements of DG_0^C can be viewed as smooth maps from an interval $[0, 1]$ parametrized by r (the radius of the disc) into the loop group LG_0^C . Since the loop group LG_0^C is diffeomorphic to the direct product of the manifolds LG_0 and $L_+G_0^C$, it follows that $\bar{\bar{l}}$ can be uniquely decomposed as

$$\bar{\bar{l}} = XA, \quad A \in DG_0, \quad X \in D_+G_0^C. \quad (7.96)$$

Here $D_+G_0^C$ is the subgroup DG_0^C consisting of the elements of the form $X(\sigma, r)$, where for every fixed $r = r_0$, $X(\sigma, r_0) \in L_+G_0^C$.

Consider a Θ^C -representative $\bar{\bar{K}} \in \mathbf{R}^2 \times_{S,Q} \widehat{DG_0^C}$ of an element $\tilde{\tilde{K}} \in \tilde{\tilde{D}}$;

$$\bar{\bar{K}} = (\bar{\bar{l}}, e^{ip+P}, w + is), \quad (7.97)$$

where $(\bar{\bar{l}}, e^{i\lambda+L}) \in \widehat{DG_0^C}$, $\bar{\bar{l}} \in DG_0^C$ and $p, P, w, s \in \mathbf{R}$. As the consequence of the decomposition (7.96), also $\bar{\bar{K}}$ can be decomposed uniquely as

$$\bar{\bar{K}} = (\bar{\bar{b}}, e^t, w) \tilde{*} (\bar{\bar{g}}, e^{i\phi}, is), \quad (7.98)$$

where $\bar{\bar{g}} \in DG_0$, $\bar{\bar{b}} \in D_+G_0^C$, $\phi, t \in \mathbf{R}$ and the product $\tilde{*}$ is that of $\mathbf{R}^2 \times_{S,Q} \widehat{DG_0^C}$.

By applying the homomorphism $\hat{\varphi}_C$ (cf. Section 4.4.3) on the both sides of (7.98), we obtain

$$\tilde{\tilde{K}} = (\hat{\varphi}_C(\bar{\bar{b}}, e^t), w) (\hat{\varphi}_C(\bar{\bar{g}}, e^{i\phi}), is). \quad (7.99)$$

Now we prove that

$$(\hat{\varphi}_C(\bar{\bar{b}}, e^t), w) = \tilde{b}_L(\tilde{\tilde{K}}), \quad (\hat{\varphi}_C(\bar{\bar{g}}, e^{i\phi}), is) = \tilde{g}_R(\tilde{\tilde{K}}). \quad (7.100)$$

This will follow immediately from the uniqueness of the decomposition $\tilde{D} = \tilde{B}\tilde{G}$, if we demonstrate that

$$(i) \quad (\hat{\varphi}_{\mathbf{C}}(\bar{\bar{b}}, e^t), w) \in \tilde{B}, \quad (ii) \quad (\hat{\varphi}_{\mathbf{C}}(\bar{\bar{g}}, e^{i\phi}), is) \in \tilde{G}. \quad (7.101)$$

The statement (ii) is the direct consequence of the way how \tilde{G} is embedded in \tilde{D} (Lemma 4.18 and its proof); the statement (i) requires a bit more of work, however. Actually we must show that it exists $b \in L_+G_0^{\mathbf{C}}$ such that

$$\hat{\varphi}_{\mathbf{C}}(\bar{b}, e^t) = \hat{\varphi}_{\mathbf{C}}(\bar{\bar{b}}, e^t). \quad (7.102)$$

Recall that \bar{b} was defined in Lemma 4.19, as the map from the *Disc* into $G_0^{\mathbf{C}}$ whose boundary is $b \in L_+G_0^{\mathbf{C}}$. Let us show that b given by

$$b = (\Pi_0^{\mathbf{C}} \circ \hat{\varphi}_{\mathbf{C}})((\bar{\bar{b}}, e^t)) \quad (7.103)$$

solves (7.102). This amounts to show that

$$C^{\mathbf{C}}(\bar{\bar{b}}\bar{b}^{-1}) = 0. \quad (7.104)$$

The last equation follows from the general fact that $C^{\mathbf{C}}$ vanishes on $D_+G_0^{\mathbf{C}} \cap \partial G^{\mathbf{C}}$. Indeed, recall that if $\bar{l} \in \delta G^{\mathbf{C}} \cap D_+G_0^{\mathbf{C}}$, then \bar{l} can be interpreted as a map from the *D*-Riemann sphere S^2 into $G_0^{\mathbf{C}}$ (cf. the discussion of the WZW term in Section 4.4 and in Appendix 7.1) and extended to a map \bar{l}_{ext} from the unit *Ball* wrapped by S^2 into $G_0^{\mathbf{C}}$. Now the map \bar{l}_{ext} can also be interpreted as a map from a certain *half-disc* into $LG_0^{\mathbf{C}}$. This half-disc is diffeomorphic to the space of orbits of the azimuthal rotation around the axis connecting the northern with the southern pole of the *Ball* and it can be clearly identified with the half-disc bounded by the north-south axis of rotation and by the half of the Greenwich Meridian. Points of each such orbit are parametrized by the loop parameter σ and the map \bar{l}_{ext} restricted to any such orbit is naturally the element of $LG_0^{\mathbf{C}}$. It is crucial to remark, that the loop group $LG_0^{\mathbf{C}}$ can be diffeomorphically decomposed as a product of LG_0 and $L_+G_0^{\mathbf{C}}$ hence also the group of smooth maps from the half-disc (with appropriate boundary conditions) into $LG_0^{\mathbf{C}}$ can be decomposed accordingly. If we decompose the map \bar{l}_{ext} in this way, we observe that the $L_+G_0^{\mathbf{C}}$ part of this decomposition is still the map that extends the original element $\bar{l} \in \partial G^{\mathbf{C}} \cap D_+G_0^{\mathbf{C}}$ into the *Ball*. We refer to such an extension as to \bar{l}_{ext}^+ and we recall that it can be understood

as the map from the half-disc into $L_+G_0^{\mathbf{C}}$. Thus we can calculate the $C^{\mathbf{C}}(\bar{l})$ by using the extension \bar{l}_{ext}^+ in the defining formula (4.132). The result is that the integral over the loop variable σ can be converted on the equatorial contour integral on the σ -Riemann sphere, where the integrated function in the z -variable is everywhere holomorphic on the southern hemisphere. The contour can be therefore shrunk to the point without encountering any singularity, hence $C^{\mathbf{C}}(\bar{l}) = 0$.

We recapitulate, what we have learned so far: having an element $\tilde{K} \in \tilde{D}$, we can find $\tilde{b}_L(\tilde{K})$ and $\tilde{g}_R(\tilde{K})$ by picking up any $\Theta^{\mathbf{C}}$ -representative $\bar{K} \in \mathbf{R}^2 \times_{S,Q} \widehat{DG}_0^{\mathbf{C}}$, decomposing \bar{K} as in (7.98) and evaluating

$$\tilde{b}_L(\tilde{K}) = (\hat{\wp}_{\mathbf{C}}(\bar{b}, e^t), w), \quad \tilde{g}_R(\tilde{K}) = (\hat{\wp}_{\mathbf{C}}(\bar{g}, e^{i\phi}), is).$$

Similarly, we can prove also that

$$\hat{b}_L(\hat{K}) = (\wp_{\mathbf{C}}(\bar{b}, 1), w), \quad \hat{g}_R(\hat{K}) = (\wp_{\mathbf{C}}(\bar{g}, e^{i\phi}), 0), \quad (7.105)$$

Here $\bar{K} = (\bar{l}, e^{ip}, w)$ is any $\Theta_{\mathbf{R}}^{\mathbf{C}}$ -representative of $\hat{K} \in \hat{D}$ with a decomposition

$$\bar{K} = (\bar{b}, 1, w) \hat{*} (\bar{g}, e^{i\phi}, 0). \quad (7.106)$$

Of course, $(\bar{l}, e^{ip}) \in {}^{\mathbf{R}}\widehat{DG}_0^{\mathbf{C}}$, $\bar{l} \in DG_0^{\mathbf{C}}$, $\bar{g} \in DG_0$, $\bar{b} \in D_+G_0^{\mathbf{C}}$, $p, w, \phi \in \mathbf{R}$ and the product $\hat{*}$ is taken in $\mathbf{R} \times_Q {}^{\mathbf{R}}\widehat{DG}_0^{\mathbf{C}}$.

Now we need the formula for $\hat{m}_L \equiv \hat{m} \circ \tilde{b}_L$. Clearly,

$$\hat{m}_L(\tilde{K}) = ((\Pi_0^{\mathbf{C}} \circ \hat{\wp}_{\mathbf{C}})(\bar{b}, e^t), w). \quad (7.107)$$

In particular, it is immediate to realize that $\hat{m}_L(\tilde{K})$ does not depend on t .

We have shown so far, how to calculate the maps $\hat{m}_L, \tilde{b}_L, \hat{b}_L, \tilde{g}_R$ and \hat{g}_R by using the "Iwasawa" decomposition (7.96) of the $\Theta^{\mathbf{C}}$ -representatives of \tilde{K} (or $\Theta_{\mathbf{R}}^{\mathbf{C}}$ -representatives of \hat{K}). We can now view the elements $\hat{k} \in \hat{G}, \hat{a} \in \hat{B}$ as elements of \hat{D} but also as elements of \tilde{D} . Their representatives can be chosen as

$$\bar{k} = (\bar{k}, e^{i\psi}, 0), \quad \bar{a} = (\bar{a}, 1, a^\infty). \quad (7.108)$$

Now it is crucial to notice, that \bar{k}, \bar{a} make sense as the elements of $\mathbf{R}^2 \times_{S,Q} \widehat{DG}_0^{\mathbf{C}}$ and also as the elements of $\mathbf{R} \times_Q {}^{\mathbf{R}}\widehat{DG}_0^{\mathbf{C}}$. If we want to calculate the

quantities $\tilde{b}_L(\widetilde{Ad_{\bar{k}}}\hat{a})$ and $\hat{b}_L(\widehat{Ad_{\bar{k}}}\hat{a})$, we should understand $\bar{\bar{k}}, \bar{a}$ respectively in the former and the latter sense.

Consider now any n elements $\bar{\bar{K}}_j, j = 1, \dots, n$ of $\mathbf{R}^2 \times_{S,Q} \widehat{DG_0^C}$ of the form

$$\bar{\bar{K}}_j = (\bar{\bar{l}}_j, e^{ip_j}, w_j). \quad (7.109)$$

As we already know, we can view $\bar{\bar{K}}_j$ of this form also as the elements of $\mathbf{R} \times_Q \mathbf{R} \widehat{DG_0^C}$. Now we denote by $\tilde{*}$ the product in $\mathbf{R}^2 \times_{S,Q} \widehat{DG_0^C}$ and by $\hat{*}$ the one in $\mathbf{R} \times_Q \mathbf{R} \widehat{DG_0^C}$. Using (4.108) and (4.129), we arrive immediately at

$$\bar{\bar{K}}_1 \tilde{*} \bar{\bar{K}}_2 \tilde{*} \dots \tilde{*} \bar{\bar{K}}_n = (1, e^{t(\bar{\bar{K}}_1, \bar{\bar{K}}_2, \dots, \bar{\bar{K}}_n)}, 0) \tilde{*} (\bar{\bar{K}}_1 \tilde{*} \bar{\bar{K}}_2 \tilde{*} \dots \tilde{*} \bar{\bar{K}}_n), \quad (7.110)$$

where l.h.s is viewed as the element of $\mathbf{R}^2 \times_{S,Q} \widehat{DG_0^C}$ and $t(., \dots, .)$ is some real function whose explicit form is not needed for the proof of the lemma. From the equation (7.110) for $\bar{\bar{K}}_1 = \bar{\bar{k}}, \bar{\bar{K}}_2 = \bar{\bar{a}}$ and $\bar{\bar{K}}_3 = \bar{\bar{k}}^{-1}$, we conclude that

$$\widehat{Ad_{\bar{\bar{k}}}\bar{\bar{a}}} = (1, e^{t(\bar{\bar{k}}, \bar{\bar{a}}, \bar{\bar{k}}^{-1})}, 0) \tilde{*} (\widehat{Ad_{\bar{\bar{k}}}\bar{\bar{a}}}). \quad (7.111)$$

Now we have the "Iwasawa" decomposition (7.106)

$$\widehat{Ad_{\bar{\bar{k}}}\bar{\bar{a}}} = (\bar{\bar{b}}, 1, w) \hat{*} (\bar{\bar{g}}, e^{i\phi}, 0), \quad (7.112)$$

for some $\bar{\bar{b}} \in D_+ G_0^C, \bar{\bar{g}} \in DG_0, \phi, w \in \mathbf{R}$. From (7.110) and (7.111), we obtain

$$\widehat{Ad_{\bar{\bar{k}}}\bar{\bar{a}}} = (\bar{\bar{b}}, e^{t'(\bar{\bar{k}}, \bar{\bar{a}})}, w) \tilde{*} (\bar{\bar{g}}, e^{i\phi}, 0), \quad (7.113)$$

where $t'(\bar{\bar{k}}, \bar{\bar{a}})$ is some real function whose form is irrelevant for us. Now from (7.100), (7.105), (7.107), (7.111), (7.112) and (7.113), we conclude immediately

$$\hat{m}_L(\widetilde{Ad_{\bar{k}}}\hat{a}) = \hat{b}_L(\widehat{Ad_{\bar{k}}}\hat{a}), \quad \tilde{g}_R(\widetilde{Ad_{\bar{k}}}\hat{a}) = \hat{g}_R(\widehat{Ad_{\bar{k}}}\hat{a}).$$

The lemma is proved.

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